論　説

# Default Timing and Recovery Rate*

Yuki Itoh†

**Abstract**

Recently, in credit risk management, quantificationing the connection between the default probability and the recovery rate on macro economics becomes one of the most important problem. However, there has been no study of the relationship between the default timing and the recovery rate of a single company. In this paper, we model the default timinig and the recovery rate for the debt of a single company in the structural model.

## 1. Introduction

### 1.1 Literature Review

Developing the quantification of the default probability and the recovery rate is the most important problem for financial institutions and their supervisors. However, the connection between the default timing and the recovery rate of a single company is rarely considered in literatures, in this paper, we analyse it.

In the theoretical model for credit risk, there are two major models: the reduced form model and the structural model. The structural model, which is developed by Merton (1974), is one of the most popular model in credit risk studies. In the Merton model, the value of a company follows a stochastic process (geometric Brownian motion in Merton (1974)) and only if the stochastic process falls below the boundary at maturity, the default of the company occurs. In the Merton model, the recovery amount of the debt if the default occurs is mainly determinated the difference the value of company and the default boundary at maturity. Black and Cox (1976) extend the Merton model and they allow the occurrence of the default before the maturity. In fact, they assume that the value of a company follows a stochastic process (geometric Brownian motion in Black and Cox (1976)) and if the stochastic process falls below the boundary in the first time, the default of the company occurs. Therefore, the Black and Cox model is also called "first passage model".

* The author is grateful for the helpful comments of Professor Junichiro Fukuchi, Department of Economics, Gakushuin University. However, all remaining errors are ours.

† Faculty of International Social Sciences, Yokohama National University; 79-4 Tokiwadai, Hodogaya-ku, Yokohama 240-8501, Japan; yitoh7@gmail.com

In empirical studies of recovery rates for the debt of the defaulting companies, the cumulative recovery rate have the bimodal distribution. For the details on the recovery rates for the debt, see Asarnow and Edwards (1995), Hurt and Felsovalyi (1998), Araten, Jacobs Jr. and Varshney (2004), Franks, de Servigny and Davydenko (2004), Dermine and de Carvalho (2006), and Itoh and Yamashita (2008).

However, existent recovery models are not explained this phenomenon. We consider that one of the main causes, which the cumulative recovery rate is distributed bimodal, may be the difference of the default timing. If the default occurs early (for example, the asset of company is more than the debt), the recovery rate may be higher. On the other side, if the default occurs late, is the recovery rate (or total loss for lender) lower?

### 1.2 Summary

In this paper, using the structural model, we model the connection between the default timing and the recovery rate. Furthermore, we analyse the optimal default point for lender.

We explain our model briefly. We assume that default occurs in the framework of structural model and lender can observe the state of company only at the discrete points in structural model. Accordingly, the default occurs only at the discrete observable points. We consider three types of the default conditions: base default model, delay default model, and early default model. In the base default model, If the ability-to-pay process becomes less than the debt, then the default occurs. The default structure of the base default model is the same as Hull and White (2001). In the delay default model, if the ability-to-pay process falls below the debt second consecutive time, then the default occurs. In the early default model, if the probability that the ability-to-pay process become less than the barrier at next observable point becomes more than a particular level, then the default occurs.

This paper is organized as follows. We introduce and derive the default probability and the recovery rate in the base default model in Section 2, those in the delay default model are given Section 3, and those in the early default model are shown in Section 4. In Section 5, we explain the numerical explain and in Section 6 we show the numerical results.

## 2. Base Default Model

In this section, we explain the base default model. Throughout this paper, we consider a single company.

**Assumption 2.1.** *The amount of the loan for the company is $D$ at time 0, and the maturity of the loan is time $T$.*

**Definition 2.2.** *Let Yt be the ability-to-pay process, which means the amount of paying the debt if the company is liquidated at time $t \in [0, T]$.*

For example, $Y_t$ is considered as the value of the company plus the value of collateral.

**Assumption 2.3.** *For $t \in [0, T]$, the ability-to-pay process follows geometric Brownian motion as follows,*

$$dY_t = \mu_Y Y_t dt + \sigma_Y Y_t dW_t. \tag{1}$$

*where $W_t$ is the Brownian motion. Moreover, we assume $Y_0 > D$.*

We assume that if the default occurs, lender recovers the debt from the value of company and the collateral. Lender requires the retirement for the debt, when the amount of the ability-to-pay falls below the face value at the latest. In this situation, the timing of observating the ability-to-pay process is important matter for the recovery. There are two cases: continuous observation and discrete observation.

First, we explain continuous observation model. Under the Assumption 2.4, the structure of default is equivalent to that of Black and Cox (1976).

**Assumption 2.4** (continuous observation)**.** *The lender can observe continuously ability-to-pay process. The default occurs, as the same time as the expected recovery falls below the face value as follows,*

$$\tau_C = \inf\{t > 0 : Y_t < D\}\ .$$

Next, we assume that the company value and the collateral value cannot be observed continuously but can be done discretely by lender. In particular, we assume that lender can only observe the company value and the collateral value at times $0 = t_0, t_1, t_2, \cdots, t_M = T$. Therefore, the default occurs discretely. Under the Assumption 2.5, the structure of default is similarly to that of Hull and White (2001). However, in Hull and White (2001), the ability-to-pay process does not follow geometric Brownian motion but it is normally distributed.

**Assumption 2.5** (discrete observation)**.** *The lender can only observe the company value and the collateral value at times* $0 = t_0, t_1, t_2, , t_M = T$. *The default occurs, when the ability-to-pay process is below the face value as follows,*

$$\tau_B = \inf\{t = t_1, t_2, \cdots, t_M : Y_t < D\}\ .$$

The expected value of the recovery rate $R$ if the default occurs in $[0, T]$ is given by,

$$E[R] = \frac{1}{D}\sum_{m=1}^{M} E\left[Y_{t_1}|\tau_B = t_1\right] P\left\{\tau_B = t_m\right\}. \tag{2}$$

**Definition 2.6.** $f_{t_m}(x)$ *is the probability that* $Y_{t_m}$ *lies* $x$ *and there has been no default prior to* $t_m$. *We call it the survival function.*

Using the transition density of geometric Brownian motion (for example, see Shreve (2004)), we have

$$f_{t_1}(x) = \frac{1}{x\sqrt{2\pi\sigma^2 t_1}}\exp\left\{-\frac{\left(\ln\frac{x}{Y_0} - \mu t_1\right)^2}{2\sigma^2 t_1}\right\},$$

$$f_{t_m}(x) = \int_D^\infty f_{t_{m-1}}(u)\frac{1}{x\sqrt{2\pi\sigma^2\left(t_m - t_{m-1}\right)}}\exp\left\{-\frac{\left(\ln\frac{x}{u} - \mu\left(t_m - t_{m-1}\right)\right)^2}{2\sigma^2\left(t_m - t_{m-1}\right)}\right\}du.$$

The probability of occurrence of the default at $t_m$ is as follows,

$$P\left\{\tau_B = t_m\right\} = \int_{B-} f_{t_m}(x)dx. \tag{3}$$

where $B_-$ is $(-\infty, D]$.

## 3. Delay Default Model

In this section, we explain the delay default model. We assume that lender can observe the ability-to-pay process only at discrete time points. Moreover, we assume that default occurs, not as soon as the ability-to-pay process falls below the debt, but the ability-to-pay process falls below the debt second consecutive time. Therefore, in the delay default model, we distinguish between insolvent (defined as the company's asset value falling below an insolvency barrier) and bankrupt (defined as legally declared inability to pay the debt).

**Assumption 3.1.** *$Y_t$ can be observed only at $M+1$ observable points $0 = t_0, t_1, t_2, \cdots, t_M = T$.*

**Assumption 3.2.** *For $t_m$, $m = 2, 3, \cdots, M-1$, the default occurs if the ability-to-pay process falls below the amount of the debt for two consecutive observable points (that is at $t_{m-1}$ and $t_m$). At $t_M$, that is maturity, the default occurs if the ability-to-pay process falls below the amount of the debt.*

In the delay default model, for $m = 2, 3, \cdots, M-1$, the default time is defined as

$$\tau_D \triangleq \inf\left\{t_m : Y_{t_{m-1}} < D, Y_{t_m} < D\right\} \wedge \{t_M : Y_{T_M} < D\}.$$

### 3.1 Survival Function and Probability of Default

In this section, we derive the survival function and the probability of default under Assumption 3.2.

**Definition 3.3.** *Let $\phi_{t_{m-1}t_m}(x_{m-1}, x_m)$ be the transition density of $Y_t$ from $x_{m-1}$ at $t_{m-1}$ to $x_m$ at $t_m$ as follows,*

$$\phi_{t_{m-1}t_m}(x_{m-1}, x_m) = P\left\{Y_{t_{m-1}} = x_{m-1}, Y_{m_k} = x_k\right\}.$$

**Definition 3.4.** *For $m = 2, 3, \cdots, M-1$, let $f_{t_{m-2}t_m}(x_m)$ is the survival function, which is the probability that the company is survival in the interval $(t_{m-2}, t_m]$ and that company value at $t_m$ is $x_m$ under the condition that the company is survival at $t_{m-2}$.*

**Definition 3.5.** *Let $B^- = (-\infty, D]$ be insolvency zone and $B^+ = (D, \infty)$ solvency zone.*

Thus, under Assumption 3.2, if the ability-to-pay process falls into $B^-$ for two consecutive observable points, the default occurs. For $m = 2, 3, \cdots, M-1$,

$$P\{Y_{t_m} = x_m, t_m > \tau\} \triangleq f_{t_{m-2}t_m}(x_m) = \begin{cases} f^+_{t_{m-2}t_m}(x_m), & x_m \in B^+, \\ f^-_{t_{m-2}t_m}(x_m), & x_m \in B^-. \end{cases} \tag{4}$$

**Proposition 3.6.** *For $m = 4, 5, \cdots, M-1$ and $x_m \in B^+$, we have*

$$\begin{aligned} f^+_{t_{m-2}t_m}(x_m) = &\iint_{B^+,B^+} f^+_{t_{m-4}t_{m-2}}(x_{m-2})\phi(x_{m-2}, x_{m-1})\phi(x_{m-1}, x_m)dx_{m-2}dx_{m-1} \\ &+ \iint_{B^+,B^-} f^+_{t_{m-4}t_{m-2}}(x_{m-2})\phi(x_{m-2}, x_{m-1})\phi(x_{m-1}, x_m)dx_{m-2}dx_{m-1} \\ &+ \iint_{B^+,B^+} f^-_{t_{m-4}t_{m-2}}(x_{m-2})\phi(x_{m-2}, x_{m-1})\phi(x_{m-1}, x_m)dx_{m-2}dx_{m-1}, \end{aligned}$$

*and for $x_m \in B^-$, we have*

$$f^-_{t_{m-2}t_m}(x_m) = \iint_{B^+,B^+} f^+_{t_{m-4}t_{m-2}}(x_{m-2})\phi(x_{m-2},x_{m-1})\phi(x_{m-1},x_m)dx_{m-2}dx_{m-1}$$
$$+ \iint_{B^+,B^-} f^-_{t_{m-4}t_{m-2}}(x_{m-2})\phi(x_{m-2},x_{m-1})\phi(x_{m-1},x_m)dx_{m-2}dx_{m-1}.$$

**Proposition 3.7.** *We have*

$$f_{t_0t_2}(x_2) = \begin{cases} f^+_{t_0t_2}(x_2), & x_2 \in B^+, \\ f^-_{t_0t_2}(x_2), & x_2 \in B^-, \end{cases}$$

*where for* $x_2 \in B^+$,

$$f^+_{t_0t_2}(x_2) = \int_{B^+} \phi(x_0,x_1)\phi(x_1,x_2)dx_1 + \int_{B^-} \phi(x_0,x_1)\phi(x_1,x_2)dx_1,$$

*and for* $x_2 \in B^-$,

$$f^-_{t_0t_2}(x_2) = \int_{B^+} \phi(x_0,x_1)\phi(x_1,x_2)dx_1.$$

**Proposition 3.8.** *We have*

$$f_{t_1t_3}(x_3) = \begin{cases} f^+_{t_1t_3}(x_3), & x_3 \in B^+, \\ f^-_{t_1t_3}(x_3), & x_3 \in B^-, \end{cases}$$

*where for* $x_3 \in B^+$,

$$f^+_{t_1t_3}(x_3) = \iint_{B^+,B^+} \phi(x_0,x_1)\phi(x_1,x_2)\phi(x_2,x_3)dx_1dx_2$$
$$+ \iint_{B^+,B^-} \phi(x_0,x_1)\phi(x_1,x_2)\phi(x_2,x_3)dx_1dx_2$$
$$+ \iint_{B^+,B^+} \phi(x_0,x_1)\phi(x_1,x_2)\phi(x_2,x_3)dx_1dx_2,$$

*and for* $x_3 \in B^-$,

$$f^-_{t_1t_3}(x_3) = \iint_{B^+,B^+} \phi(x_0,x_1)\phi(x_1,x_2)\phi(x_2,x_3)dx_1dx_2$$
$$+ \iint_{B^+,B^-} \phi(x_0,x_1)\phi(x_1,x_2)\phi(x_2,x_3)dx_1dx_2.$$

If we know $x_0$ which is the value of the ability-to-pay at time 0, we can calculate $f_{t_{m-1}t_m}(x_m)$, $m = 4, 5, \cdots, M-1$ from Proposition 3.6, Proposition 3.7, and Proposition 3.8.

## 3.2 Recovery Rate

Next, we derive the recovery size, if the default occurs at $t_m$.

**Proposition 3.9.** *For* $x_2 \in B^-$, *we have*

$$P\{Y_{t_2} = x_2, \tau_D = t_2\} = \int_{B^-} \phi_{t_0t_1}(x_0,x_1)\phi_{t_1t_2}(x_1,x_2)dx_1.$$

*For* $m = 3, 4, \cdots, M-1$ *and* $x_m \in B^-$, *we have*

$$P\left\{Y_{t_m}=x_m,\tau_D=t_m\right\}=\int_{B^-}f^-_{t_{m-1}}(x_{m-1})\phi(x_{m-1},x_m)dx_{m-1}.$$

*Further, we have*

$$P\left\{Y_{t_M}=x_M,\tau_D=t_M\right\}=f^-_{t_M}(x_M).$$

Next, we derive the expected value of the recovery. If default occurs, expected value of the recovery is

$$\begin{aligned}E\left[Y_{\tau_D}|\tau_D<T\right]&=\sum_{m=2}^{M}E\left[Y_{t_m}|\tau_D=t_m\right]P\left\{\tau_D=t_m\right\}\\&=\sum_{m=2}^{M}E\left[Y_{t_m},\tau_D=t_m\right].\end{aligned}$$

**Proposition 3.10.** *We have*

$$E\left[Y_2,\tau_D=t_2\right]=\iint_{B^-,B^-}x_2\phi_{t_0t_1}(x_0,x_1)\phi_{t_1t_2}(x_1,x_2)dx_1dx_2,$$

*for* $m=3,4,\cdots,M-1$,

$$E\left[Y_{t_m},\tau_D=t_m\right]=\iint_{B^-,B^-}x_mf^-_{t_m}(x_{m-1})\phi(x_{m-1},x_m)dx_{m-1}dx_m,$$

*and*

$$E\left[Y_{t_M},\tau_D=t_M\right]=\int_{B^-}f^-_{t_M}(x_M)dx_M.$$

### 3.3 Probability of Default

In this section, we derive the probability of default. The probability function of default time is as follows, for $m=2$,

$$\begin{aligned}P\left\{\tau_D=t_2\right\}&=P\left\{Y_{t_1}<D,Y_{t_2}<D\right\}\\&=\int_0^D\int_0^D\phi_{t_0t_1}(x_0,x_1)\phi_{t_1t_2}(x_1,x_2)dx_1dx_2,\end{aligned}$$

and for $m=3,4,\cdots,M-1$,

$$P\left\{\tau_D=t_m\right\}=P\left\{Y_{t_{m-1}}<D,Y_{t_m}<D|\tau_D>t_{m-1}\right\}. \tag{5}$$

We derive (5) the probability of default at $t_m$.

**Proposition 3.11.** *For* $m=3,4,\cdots,M-1$, *the probability of default at* $t_m$ *is*

$$P\left\{\tau_D=t_m\right\}=\iint_{B^-,B^-}f^-_{t_{m-1}}(x_{m-1})\phi(x_{m-1},x_m)dx_{m-1}dx_m.$$

*Moreover, the probability of default at* $t_M$ *is*

$$P\left\{\tau_D=t_M\right\}=\int_{B^-}f^-_{t_M}(x_M)dx_M.$$

## 4. Early Default Model

In this section, we consider earlier default than base model. For example, the lender can claim the early redeem of the debt by the safety covenants.

**Assumption 4.1.** *$Y_t$ can be observed only at $M+1$ observable points $0 = t_0, t_1, t_2, \cdots, t_M = T$.*

**Assumption 4.2.** *For $t_m$, $m = 1, 2, \cdots, M-1$, the default occurs, as the same time as $Y_t$ falls below the face value. In addition, for $t_m$, $m = 1, 2, \cdots, M-1$ if the company is suravival at $t_{m-1}$ and $Y_{t_m} > D$, the default occurs as follows as soon as the probability, that the ability-to-pay process becomes the below $D$ at $t_{m+1}$, is less than $\alpha$, the default is occurs where $0 < \alpha < 1$. At $t_M$, that is maturity, the default occurs if the ability-to-pay process is below the amount of the debt.*

Under Assumption 4.2, for $m = 1, 2, \cdots, M-1$, the default time is defined as

$$\tau_E = \inf\left\{t_m : Y_{t_m} < D^*_{t_m}\right\} \wedge \left\{t_M : Y_{T_M} < D\right\},$$

such that $D^*(x_{t_m}) > D$ meets

$$\int_{-\infty}^{D} \phi_{x_{t_m}, x_{t_{m+1}}}\left(D^*_{t_m}, x\right) dx = \alpha. \tag{6}$$

Similarly as (2), if the default occurs in $[0, T]$, the recovery rate is given by,

$$E[R_E] = \frac{1}{D}\sum_{m=1}^{M} E\left[Y_{t_m}|\tau_E = t_m\right] P\left\{\tau_E = t_m\right\}. \tag{7}$$

## 5. Numerical Experiments

In this section, we explain the methods of calculating the probability distribution of the state of the company at default.

### 5.1 Base Default Model

For the purpose of calculating the state of company at default, if we calculate (3) straight, the multiple integral is required many computational time in the case where the observation points increase. Therefore, we discretize the increment of the ability-to-pay process in order to use the probability transition matrix.

**Assumption 5.1.** *We discretize $Y_t$ by the rounding method, using right endpoint in span. We set upper bound of $Y_t$ for convenience. For $t = t_1, t_2, \cdots, t_M$, we assume $Y_t \in \{y_1, \cdots, y_J\}$ where $y_J$ is upper bound of $Y_t$.*

For the details on the rounding method, see Klugman, Panjer and Willmot (2004) and Itoh (2008).

**Definition 5.2.** *Let $y_d = D$ be the debt value, $^{\exists}d \in [1, J]$. It is the insolvency barrier.*

**Definition 5.3.** *Let $T_m = t_m - t_{m-1}$. For $m = 1, 2, \cdots, M$ and $(i, j) \in \{(1, 1), (1, 2), \cdots, (J, J)\}$, let $q_{y_i}^{Tm}(y_j)$ be the probability transition density from $y_i$ at $t_{m-1}$ to $y_j$ at $t_m$. Moreover, let $\mathbf{Q}_{Tm}$ be the transition matrix from $t_{m-1}$ to $t_m$ as follows,*

$$\boldsymbol{Q}_{T_m} = \begin{pmatrix} q_{y_1}^{T_m}(y_1) & q_{y_1}^{T_m}(y_2) & \cdots & q_{y_1}^{T_m}(y_J) \\ q_{y_2}^{T_m}(y_1) & q_{y_2}^{T_m}(y_2) & \cdots & q_{y_2}^{T_m}(y_J) \\ \vdots & \vdots & \ddots & \vdots \\ q_{y_J}^{T_m}(y_1) & q_{y_J}^{T_m}(y_2) & \cdots & q_{y_J}^{T_m}(y_J) \end{pmatrix}. \tag{8}$$

From Assumption 2.3 and Assumption 5.1, for $m = 1, 2, \cdots, M$, we obtain the probability transition density as follows,

$$q_{y_i}^{T_m}(y_j) = \begin{cases} \int_{y_j}^{\infty} \frac{1}{x\sqrt{2\pi\sigma^2 T_m}} \exp\left\{ -\frac{\left(\ln \frac{x}{y_i} - \mu T_m\right)^2}{2\sigma^2 T_m} \right\} dx, & i > d, j = J, \\ \int_{y_{j-1}}^{y_j} \frac{1}{x\sqrt{2\pi\sigma^2 T_m}} \exp\left\{ -\frac{\left(\ln \frac{x}{y_i} - \mu T_m\right)^2}{2\sigma^2 T_m} \right\} dx, & i > d, j < J, \\ 1, & i \le d, i = j, \\ 0, & i \le d, i \ne j, \end{cases}$$

where $y_0 = 0$. Then, for $m = 1, 2, \cdots, M-1$, (8) is transformed into

$$\boldsymbol{Q}_{T_m} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ q_{y_{d+1}}^{T_m}(y_1) & q_{y_{d+1}}^{T_m}(y_2) & \cdots & q_{y_{d+1}}^{T_m}(y_J) \\ \vdots & \vdots & \ddots & \vdots \\ q_{y_J}^{T_m}(y_1) & q_{y_J}^{T_m}(y_2) & \cdots & q_{y_J}^{T_m}(y_J) \end{pmatrix}.$$

If the initial debt amount is $y_i$, the probability transition vector at $t_0$ is given by

$$\mathcal{P}_0 = (0, \cdots, 0, 1, 0, \cdots 0),$$

thus, the column vector whose $i$-th element is 1 and whose other elements are 0. Then, we have the probability transition vector at $t_M$ as follows,

$$\mathcal{P}_{t_M} = \mathcal{P}_0 \boldsymbol{Q}_{T_1} \boldsymbol{Q}_{T_2} \cdots \boldsymbol{Q}_{T_M}. \tag{9}$$

## 5.2 Delay Default Model

In this section, we present the probability transition matrix and probability transition vector in the delay default model.

In the base default model, if the ability-to-pay process falls the less or equal to the boundary, the default occurs and then the state of company is the only one (that is bankruptcy). In the delay default model, though the ability-to-pay process falls the less or equal to the boundary, the company may be in operation. In this section, we calculate the transition vector of the modified ability-to-pay process, which is the ability-to-process in the delay default model.

**Definition 5.4.** *Let $\hat{Y}$ be the modified ability-to-pay process. For $t = t_1, t_2, \cdots, t_{M-1}$, we assume $\hat{Y}_t \in \{y_1, \tilde{y}_1, y_2, \tilde{y}_2, \cdots, y_k, \tilde{y}_k, y_{k+1}, \cdots, y_J\}$. $y_i$ is the state that company is survived and that the amount of ability- to-pay is $y_i$. $\tilde{y}_i$ is the state that the company is default and that the amount of ability-to-pay is $y_i$. Moreover, let $\mathbf{Q}_{T_m}$ be the transition matrix from $t_{m-1}$ to $t_m$ as follows,*

$$
Q_{T_m}=\begin{pmatrix}
q_{y_1}(y_1) & q_{y_1}(\tilde{y}_1) & q_{y_1}(y_2) & q_{y_1}(\tilde{y}_2) & \cdots & q_{y_1}(y_d) & q_{y_1}(\tilde{y}_d) & q_{y_1}(y_{d+1}) & \cdots & q_{y_1}(y_J)\\
q_{\tilde{y}_1}(y_1) & q_{\tilde{y}_1}(\tilde{y}_1) & q_{\tilde{y}_1}(y_2) & q_{\tilde{y}_1}(\tilde{y}_2) & \cdots & q_{\tilde{y}_1}(y_d) & q_{\tilde{y}_1}(\tilde{y}_d) & q_{\tilde{y}_1}(y_{d+1}) & \cdots & q_{\tilde{y}_1}(y_J)\\
q_{y_2}(y_1) & q_{y_2}(\tilde{y}_1) & q_{y_2}(y_2) & q_{y_1}(\tilde{y}_2) & \cdots & q_{y_2}(y_d) & q_{y_2}(\tilde{y}_d) & q_{y_2}(y_{d+1}) & \cdots & q_{y_2}(y_J)\\
q_{\tilde{y}_2}(y_1) & q_{\tilde{y}_2}(\tilde{y}_1) & q_{\tilde{y}_2}(y_2) & q_{\tilde{y}_2}(\tilde{y}_2) & \cdots & q_{\tilde{y}_2}(y_d) & q_{\tilde{y}_2}(\tilde{y}_d) & q_{\tilde{y}_2}(y_{d+1}) & \cdots & q_{\tilde{y}_2}(y_J)\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
q_{y_d}(y_1) & q_{y_d}(\tilde{y}_1) & q_{y_d}(y_2) & q_{y_d}(\tilde{y}_2) & \cdots & q_{y_d}(y_d) & q_{y_d}(\tilde{y}_d) & q_{y_d}(y_{d+1}) & \cdots & q_{y_d}(y_J)\\
q_{\tilde{y}_d}(y_1) & q_{\tilde{y}_d}(\tilde{y}_1) & q_{\tilde{y}_d}(y_2) & q_{\tilde{y}_d}(\tilde{y}_2) & \cdots & q_{\tilde{y}_d}(y_d) & q_{\tilde{y}_d}(\tilde{y}_d) & q_{\tilde{y}_d}(y_{d+1}) & \cdots & q_{\tilde{y}_d}(y_J)\\
q_{y_{d+1}}(y_1) & q_{y_{d+1}}(\tilde{y}_1) & q_{y_{d+1}}(y_2) & q_{y_{d+1}}(\tilde{y}_2) & \cdots & q_{y_{d+1}}(y_d) & q_{y_{d+1}}(\tilde{y}_d) & q_{y_{d+1}}(y_{d+1}) & \cdots & q_{y_{d+1}}(y_J)\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots\\
q_{y_J}(y_1) & q_{y_J}(\tilde{y}_1) & q_{y_J}(y_2) & q_{y_J}(\tilde{y}_2) & \cdots & q_{y_J}(y_d) & q_{y_J}(\tilde{y}_d) & q_{y_J}(y_{d+1}) & \ddots & q_{y_J}(y_J)
\end{pmatrix}.
$$

From Assumption 3.2, for $(i, j) \in \{(1, 1), (1, 2), \cdots, (J, J)\}$, the transition density of $\hat{Y}$ is obtain by,

$$
\begin{aligned}
q_{\tilde{y}_i}(\tilde{y}_j) &= \begin{cases} 1, & i=j, i\le d, j\le d,\\ 0, & i\ne j, i\le d, j\le d,\end{cases}\\
q_{y_i}(y_j) &= \begin{cases} 0, & i\le d, j\le d,\\ \int_{y_{j-1}}^{y_j}\phi(x)dx, & \text{otherwise},\end{cases}\\
q_{y_i}(\tilde{y}_j) &= \begin{cases} \int_{y_{j-1}}^{y_j}\phi(x)dx, & i\le d, j\le d,\\ 0, & i> d, j\le d,\end{cases}\\
q_{\tilde{y}_i}(y_j) &= 0, \quad i\le d.
\end{aligned}
$$

Therefore, we have

$$
Q_{T_m}=\begin{pmatrix}
0 & q_{y_1}(\tilde{y}_1) & 0 & q_{y_1}(\tilde{y}_2) & \cdots & 0 & q_{y_1}(\tilde{y}_d) & q_{y_1}(y_{d+1}) & \cdots & q_{y_1}(y_J)\\
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
0 & q_{y_2}(\tilde{y}_1) & 0 & q_{y_2}(\tilde{y}_2) & \cdots & 0 & q_{y_2}(\tilde{y}_d) & q_{y_2}(y_{d+1}) & \cdots & q_{y_2}(y_J)\\
0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & q_{y_d}(\tilde{y}_1) & 0 & q_{y_d}(\tilde{y}_2) & \cdots & 0 & q_{y_d}(\tilde{y}_d) & q_{y_d}(y_{d+1}) & \cdots & q_{y_d}(y_J)\\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0\\
q_{y_{d+1}}(y_1) & 0 & q_{y_{d+1}}(y_2) & 0 & \cdots & q_{y_{d+1}}(y_d) & 0 & q_{y_{d+1}}(y_{d+1}) & \cdots & q_{y_{d+1}}(y_J)\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots\\
q_{y_J}(y_1) & 0 & q_{y_J}(y_2) & 0 & \cdots & q_{y_J}(y_d) & 0 & q_{y_J}(y_{d+1}) & \ddots & q_{y_J}(y_J)
\end{pmatrix}.
$$

Similarly as (9), the probability transition vector at $t_M$ is given by,

$$
\mathcal{P}_{t_M} = \mathcal{P}_0 \boldsymbol{Q}_{T_1}\boldsymbol{Q}_{T_2}\cdots\boldsymbol{Q}_{T_M}.
$$

### 5.3 Early Default Model

In early default model, it is important to calculate the $D^*$ in (6). Using the least squares approach, we numerically calculate $D^*$ as follows,

$$
\hat{D}^* = \underset{D^*}{\operatorname{argmin}}\left(\int_0^D \phi_{x_{t_m},x_{t_{m+1}}}(D^*,x)\,dx-\alpha\right)^2. \tag{10}
$$

Obtaining the $\hat{D}^*$, we use “optim” command and “BFGS” option in R. The remaining calculation methods are the same as those of the base default model.

## 6. Numerical Results

In this section, we show the numerical results. For all calculation, we use R version 2.8.1. The probability distributions of company value in base, delay, and early default model are shown by Figures 1, 2, and 3, respectively. The default probability, the expected cumulative recovery rates, and the expected cumulative loss rates of three models is shown 4, 5, and 6. The parameters are $\mu = 0.05$, $\sigma = 0.2$, $y_0 = 12$, $D = 10$, and the length between the two observable points is 0.05 and the parameter of the early default model $\alpha$ is 0.3.

## References

Araten, Michel, Michael Jacobs Jr., and Peeyush Varshney (2004) "Measuring LGD on Commercial Loans: An 18-Year Internal Study," *The RMA Journal*, Vol. 86, No. 8, pp. 96–103.

Asarnow, E. and D. Edwards (1995) "Measuring Loss on Defaulted Bank Loans: A 24-Year Study," *The Journal of Commercial Lending*, Vol. 77, No. 7, pp. 11–23.

Black, Fischer and John C. Cox (1976) "Valuing corporate securities: Some effects in bond indenture provisions," *Journal of Finance*, Vol. 31, No. 2, pp. 351–367.

Dermine, Jean and Cristina Neto de Carvalho (2006) "Bank loan losses-given-default: A case study," *Journal of Banking & Finance*, Vol. 30, pp. 1219–1243.

Franks, Julian, Arnaud de Servigny, and Sergei Davydenko (2004) "A Comparative Analysis of the Recovery Process and Recovery Rates for Private Companies in the UK, France and Germany,"Technical report, Standard and Poor's Risk Solutions.

Hull, John and Alan White (2001) "Valuing Credit Default Swaps II: Modeling Default Correlations," *Journal of Derivatives*, Vol. 8, No. 3, pp. 12–22.

Hurt, Lew and Akos Felsovalyi (1998) "Measuring Loss on Latin American defaulted Bank Loans: A 27-Year Study of 27 Countries," *Journal of Lending & Credit Risk Management*, Vol. 80, pp. 41–46.

Itoh, Yuki (2008) "Recovery Process Model," *Asia-Pacific Financial Markets*, Vol. 15, No. 3-4, pp. 307–347.

Itoh, Yuki and Satoshi Yamashita (2008) "Empirical Study of Recovery Rates of the Loans for Small Company," *FSA Research Review*, Vol. 2007, pp. 189–218, in Japanese.

Klugman, Stuart A., Harry H. Panjer, and Gordon E. Willmot (2004) *Loss models : from data to decisions*: Wiley-Interscience.

Merton, Robert C. (1974) "On the Pricing of Corporate Debt: the risk structure of interest rates," *Journal of Finance*, Vol. 29, No. 2, pp. 449–470.

Shreve, Steven E. (2004) *Stochastic Calculus for Finance II Continuous-Time Models*: Springer.

〔Yuki Itoh, Associate Professor, Faculty of International Social Sciences, Yokohama National University〕

〔2019年 6 月18日受理〕

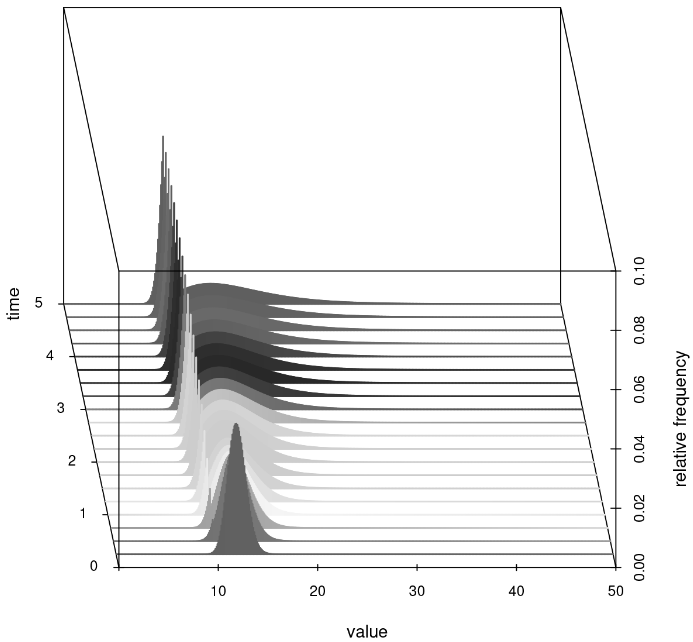


**Figure 1: Probability distribution of company's value in base model, $\mu = 0.05$, $\sigma = 0.2$, $Y_0 = 12$, $D = 10$.**

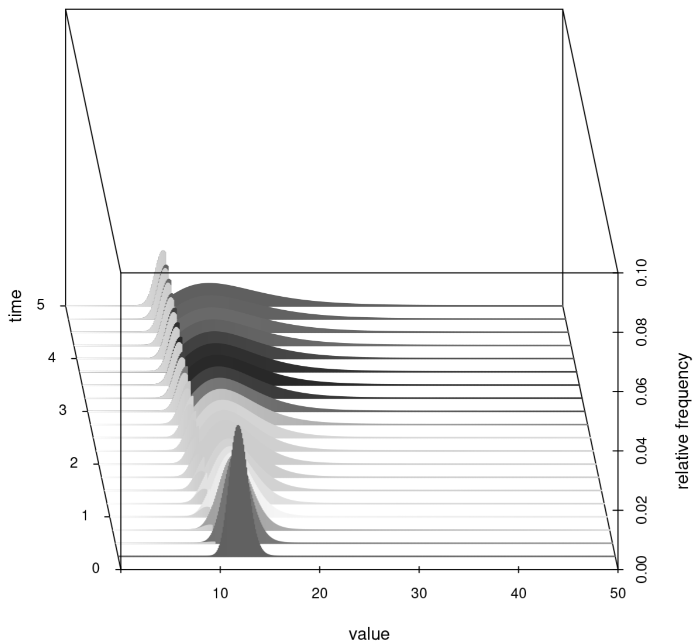


**Figure 2: Probability distribution of company's value in delay default model, $\mu = 0.05$, $\sigma = 0.2$, $Y_0 = 12$, $D = 10$. Grey area is the proba- bility distribution of company's value in default.**

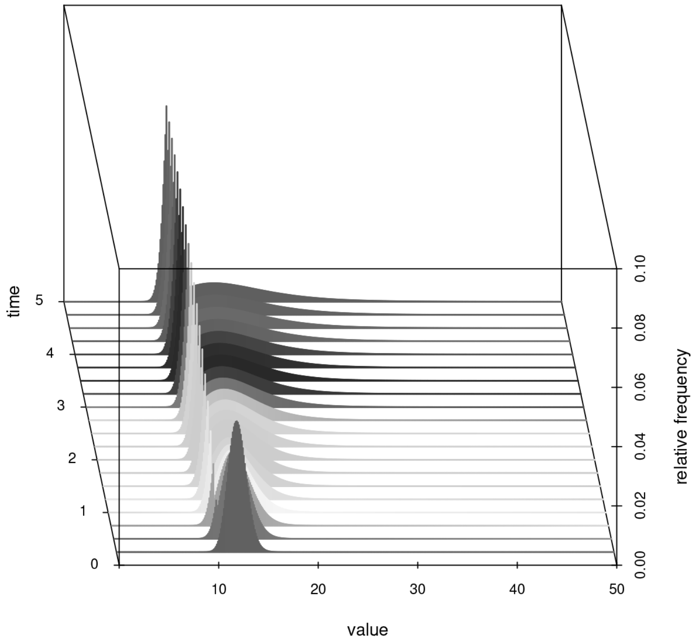


**Figure 3: Probability distribution of company's value in early default model, $\mu = 0.05$, $\sigma = 0.2$, $Y_0 = 12$, $D = 10$, $\alpha = 0.3$.**

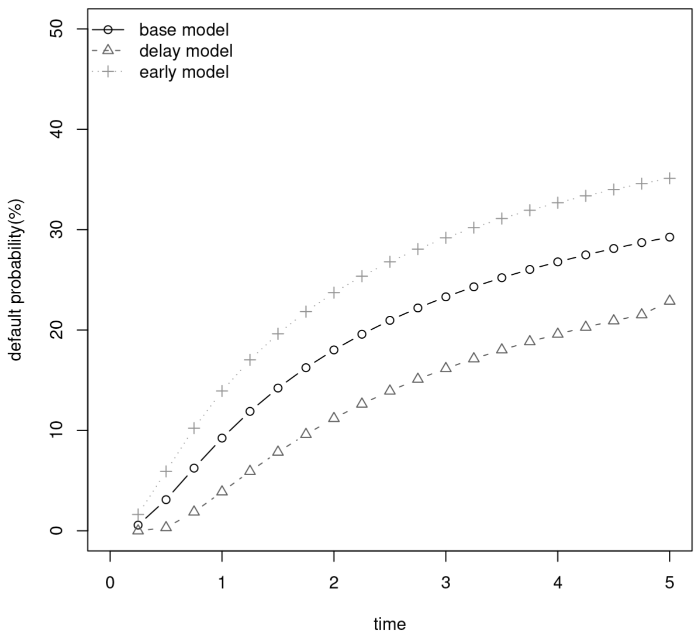


**Figure 4: Default probability, $\mu = 0.05$, $\sigma = 0.2$, $Y_0 = 12$, $D = 10$, $\alpha = 0.3$.**

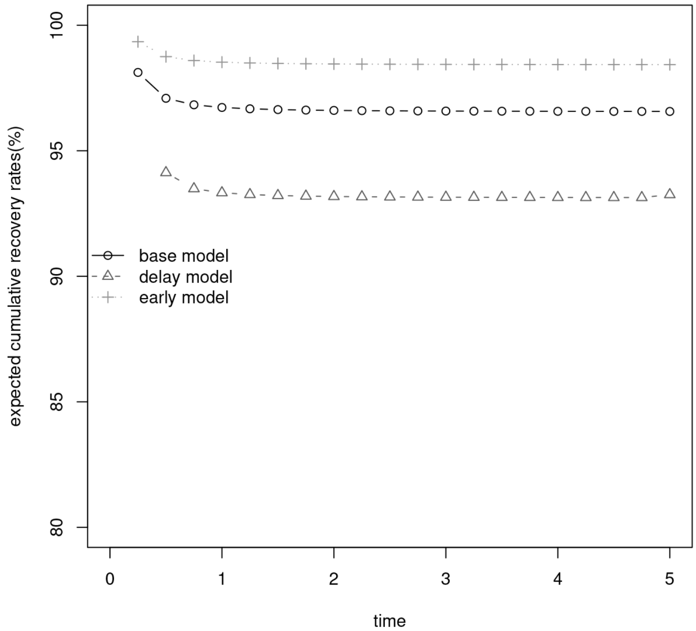


**Figure 5: Expected cumulative recov- ery rates, $\mu = 0.05$, $\sigma = 0.2$, $Y_0 = 12$, $D = 10$, $\alpha = 0.3$.**

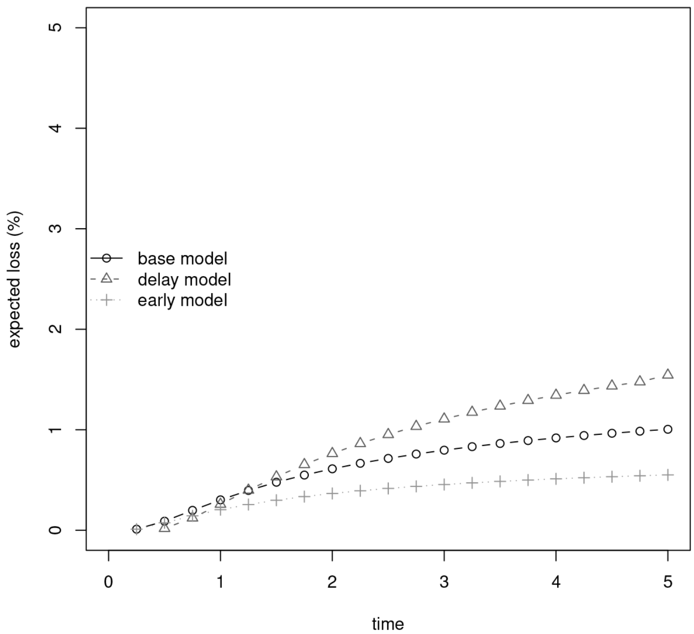


**Figure 6: Expected cumulative loss rates, $\mu = 0.05$, $\sigma = 0.2$, $Y_0 = 12$, $D = 10$, $\alpha = 0.3$.**