論　説

Possibility Relation Systems

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Abstract: A possibility distribution is regarded as a knowledge representation. The measure of ignorance and fuzziness of a possibility distribution are defined by a normality factor and the area of a possibility distribution, respectively. A rule of combination of possibility distributions is given. Furthermore, possibilistic relational systems are represented as a joint possibility distribution of relationships between inputs and outputs.These challenging issues would be a part of evidence theory dealing with expert knowledge.

1　Introduction

The remarkable advance of computer techniques has brought about a present-day information age characterized by the acceleration, intellectualization and globalization of information, which has stimulated a more emergent requirement for dealing with the huge and sophisticated information in the real world. Knowledge representation and decision based on possibility theory is one of newly-emerging information techniques to intelligently deal with human knowledge for meeting such needs [2－3, 7, 10－11].

Generally speaking the vagueness and ambiguity of human understanding, the ignorance of cognition and the diversity of evaluation are always contained in human knowledge. A possibility distribution is a kind of representation of knowledge and information where the center reflects the most possible case and the spread reflects the others with relatively low possibilities.

This paper is devoted to the properties of exponential possibility distributions in which a rule of combination of distributions similar to Dempster's rule [1] and fuzzy relation systems [8, 12－15] are considered. Since possibility distributions can be identified from numerical data associated with the possibility grades given by experts' knowledge [4－5], a possibility distribution is regarded as a knowledge representation. The measure of ignorance and fuzziness of a possibility distribution are defined by a normality factor and the area of a possibility distribution, respectively. The measure of ignorance is similar to the weight of conflict by Shafer [9], and the measure of fuzziness is the same as one defined by Kaufman and Gupta [6]. Furthermore, possibilistic relational systems are represented as a joint possibility distribution of relationships between inputs and outputs. It could be said that these challenging issues would be a part of evidence theory dealing with expert knowledge.

The paper is organized as follows. In Section 2, the most common axiomatic characterizations of possibility theory and some basic properties of possibility and necessity measures are introduced. In Section 3, Combination rule of exponential possibility distributions are introduced. In Section 4, possibility relation systems are addressed. Finally, concluding remarks for this research are made in Section 5.

2 Preliminaries of possibility theory

Possibility theory is one of the current uncertainty theories devoted to handling of incomplete information in the real world. Possibility theory has some relation with probability measure and fuzzy measure.

Given a universal set Ω and a σ-field Γ on Ω, probability measure *f* is the mapping as follows

$$f:\Gamma \to [0,1],$$

that satisfy the following requirements:

(F1) $P(\Phi)=0$

(F2) $P(\Omega)=1;$

(F3) For any $A_i \in \Gamma$ and $A_j \in \Gamma$, if

$$A_i \cap A_j = \Phi\ (i \neq j, i, j=1,2,\ldots,) \to P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \text{ (Additivity).}$$

A fuzzy measure *g* is a mapping as follows

$$g:\Gamma \to [0,1]$$

that satisfies the following requirements:

(G1) $g(\Phi)=0$;

(G2) $g(\Omega)=1$;

(G3) for all A and $\mathrm{B} \in \Gamma$, if $A \subseteq B$, then $g(A) \leq g(B)$ (monotonicity).

It can be seen that fuzzy measure is a generalization of probability measure for dealing with the non-additivity cases where the additivity is loosen to be monotonicity. Possibility theory is based on two dual fuzzy measures-possibility measure and necessity measure defined below.

A possibility measure, *Pos*, is a function

$$Pos:\Gamma \to [0,1]$$

that satisfies the following requirements:

(Pos1) $Pos(\Phi)=0$;

(Pos2) $Pos(\Omega)=1$;

(Pos3) for any family $\{A_i \mid A_i \in \Gamma, i \in I\}$, where I is an arbitrary index set, $pos\left(\bigcup_{i \in I} A_i\right) = \sup_{i \in I} Pos(A_i)$.

A necessity measure, *Nec*, is a function

$$Nec:\Gamma \to [0,1]$$

that satisfies the following requirements

(Nec1) $Nec(\Phi)=0$;

(Nec2) $Nec(\Omega)=1$;

(Nec3) for any family $\{A_i | A_i \in \Gamma, i \in I\}$, where I is an arbitrary index set, $Nec\left(\bigcap_{i\in I} A_i\right) = \inf_{i\in I} Nec(A_i)$.

It can be seen that the possibility measure is the lower semicontinuous fuzzy measure (for any increasing sequence $A_1 \subseteq A_2 \subseteq \cdots$ of sets in Γ, if $\bigcup_{i=1}^{\infty} A_i \in \Gamma$, then $\lim_{i\to\infty} g(A_i) = g\left(\bigcup_{i=1}^{\infty} A_i\right)$) and the necessity measure is the upper semicontinuous fuzzy measure (for any decreasing sequence $A_1 \supseteq A_2 \supseteq \cdots$ of sets in Γ, if $\bigcap_{i=1}^{\infty} A_i \in \Gamma$, then $\lim_{i\to\infty} g(A_i) = g\left(\bigcap_{i=1}^{\infty} A_i\right)$).

The dual relation between possibility measure and necessity measure holds as

$$Nec(A) = 1 - Pos(A^c), \tag{1}$$

which means that based on the formula (1), Giving either of the definitions of possibility measure and necessity measure can lead to the other. It is obvious that based on (1), (Pos1) and (Pos2) can lead to (Nec2) and (Nec1), respectively. Let us now check the case of $A = \bigcap_{i\in I} A_i$. In this case, $1-Pos(A^c) = 1 - Pos\left(\left(\bigcap_{i\in I} A_i\right)^c\right) = 1 - Pos\left(\bigcup_{i\in I} A_i^c\right) = 1 - \sup_{i\in I} Pos(A_i^c) = \inf_{i\in I}\left(1 - Pos(A_i^c)\right)$ holds from (Pos3), $\inf_{i\in I}\left(1 Pos(A_i^c)\right) = \inf_{i\in I} Nec(A_i)$ and $1 - Pos(A^c) = Nec(A)$ hold from (1) so that $Nec\left(\bigcap_{i\in I} A_i\right) = \inf_{i\in I} Nec(A_i)$ that is (Nec3). Likewise, it is true that based on (1), (Nec1), (Nec2) and (Nec3) can lead to (Pos1), (Pos2) and (Pos3).

Definition 1. Given a function

$$r: X \to [0,1], \tag{2}$$

if

$$\sup_{x\in X} r(x) = 1, \tag{3}$$

then the function r is called the possibility distribution of X.

It can be seen that the possibility distribution characterizes the unique possibility and necessity measure via the following formulas

$$Pos(A) = \sup_{x\in A\in\Gamma(X)} r(x), \tag{4}$$

$$Nec(A) = 1 - \sup_{x\in A\in\Gamma(X)} r(x), \tag{5}$$

$$r(x) = Pos(\{x\}) \quad x \in A. \tag{6}$$

Give a possibility distribution $\Pi_A(x)$ and a fuzzy event (fuzzy set) B with the membership function $\mu_B(x)$. The definitions of possibility and necessity measures of B based on $\Pi_A(x)$ are as follows.

$$\Pi_A(B) = \sup_x \{\Pi_A(x) \wedge \mu_B(x)\}, \tag{7}$$

$$N_A(B) = \inf_x \{(1 - \Pi_A(x)) \vee \mu_B(x)\}, \tag{8}$$

Similarly, the following dual relation between $\Pi_A(B)$ and $N_A(B)$ holds

$$N_A(B) = 1 - \Pi_A(B^c). \tag{9}$$

(9) can be easily understand from the following transformation.

$$1 - \Pi_A(B^c) = 1 - \sup_x \left\{\Pi_A(x) \wedge (1 - \mu_B(x))\right\}$$
$$= \inf_x \left\{1 - \Pi_A(x) \wedge (1 - \mu_B(x))\right\} = \inf_x \left\{(1 - \Pi_A(x)) \vee \mu_B(x)\right\} = N_A(B), \tag{10}$$

where $1 - a \wedge b = (1-a) \vee (1-b)$ is used.

Let X be a possibilistic variable governed by a possibility distribution π_A. Given an inequality relation

$$X \le z, \tag{11}$$

the possibility and necessity measures of $X \le z$ denoted as $Pos(X \le z)$ and $Nes(X \le z)$, respectively, are obtained from (4) and (5) as follows:

$$Pos(X \le z) = \sup\left\{\pi_A(x) \mid x \le z\right\}, \tag{12}$$
$$Nes(X \le z) = 1 - \sup\left\{\pi_A(x) \mid x > z\right\}. \tag{13}$$

In the cases of $Pos(X \le z)$ and $Nes(X \le z)$, B is the crisp set $(-\infty, z]$.

3　Combination Rule of Exponential Possibility Distributions

An exponential possibility distribution is regarded as a representation of evidence, which is represented by an exponential function as follows:

$$\Pi_A(x) = \exp\left\{-(x-a)^t D_A (x-a)\right\}, \tag{14}$$

where the evidence is denoted as A, a is a center vector and D_A is a symmetrical positive definite matrix. The parametric representation of A is written as follows

$$\Pi_A = \left(a, D_A^{-1}\right)_e. \tag{15}$$

$\Pi_A(x)$ is normal, that is, there is an x such that $\Pi_A(x) = 1$. Let us assume that A' is not normal. Thus, A' is given as

$$\Pi_A(x) = c \exp\left\{-(x-a)^t D_A (x-a)\right\}, \tag{16}$$

where 0<c<1.

Definition 2. Let a measure of ignorance of A' denoted as $I(A')$ be defined by

$$I(A') = -\log c. \tag{17}$$

It can be seen from Definition 2 that the possibility distribution given by (14) has no ignorance. Thus, given the evidence A' expressed by (16), A' should be normalized to obtain a normal possibility A with $I(A)$, i.e.,

$$\Pi_A(x) = \Pi_{A'}(x) / c, \tag{18}$$
$$I(A) = -\log c. \tag{19}$$

Thus, it should be noted that the given evidence A' is represented by $\Pi_A(x)$ with $I(A)$.

Definition 3. Let a measure of fuzziness of A denoted as $H(A)$ be defined as

$$H(A)=\int_{-\infty}^{\infty}\exp\left\{-(x-a)^t D_A(x-a)\,dx\right. . \tag{20}$$

The characteristic of an evidence A can be represented as

$$\left\{\left(a,D_A^{-1}\right)_e, I(A), H(A)\right\}. \tag{21}$$

Theorem 1. $H(A)$ can be rewritten as

$$H(A)=\pi^{n/2}\left|D_A^{-1}\right|^{1/2}, \tag{22}$$

Proof. Integrating a normal probability distribution from $-\infty$ to $+\infty$, its value is one. Thus, we have

$$H(A)=(2\pi)^{n/2}\left|(2D_A)^{-1}\right|^{1/2}=\pi^{n/2}\left|D_A^{-1}\right|^{1/2}. \tag{23}$$

Thus, (22) is proved. □

Let us denote a positive definite matrix D_A as $D_A>0$ and a semi-positive definite matrix D_A as $D_A\geq 0$. Also, $D_A\geq D_B$ means $D_A-D_B\geq 0$.

Theorem 2. If $D_A\geq D_B>0$,

$$H(A)\leq H(B), \tag{24}$$

where $A=\left(a,D_A^{-1}\right)_e$ and $B=\left(b,D_B^{-1}\right)_e$.

Proof. If $D_A\geq D_B>0$, $|D_A|\geq|D_B|$ holds, and also $D_B^{-1}\geq D_A^{-1}>0$ holds. It follows from this fact that Theorem 2 holds. □

Let us define a combination rule of possibility distributions from a similar view to Dempster's rule [1].

Definition 4. Let $A_1\oplus A_2$ denote the combination of possibility distributions $A_1=\left(a_1,D_1^{-1}\right)_e$ and $A_2=\left(a_2,D_2^{-1}\right)_e$. Then the combination rule is defined as,

$$A_1\oplus A_2=k\Pi_{A_1}\cdot\Pi_{A_2}, \tag{25}$$

where k is a normalizing factor such that

$$\max_x A_1\oplus A_2=1. \tag{26}$$

It is clear from Definition 2 that the measure of ignorance of $A_1\oplus A_2$ is given by

$$I(A_1\oplus A_2)=\log k, \tag{27}$$

which is similar to the measure of conflict defined by shafer [9]. Thus, $I(A_1\oplus A_2)$ can be regarded as the measure of conflict between A_1 and A_2.

Theorem 3. Let $A_1 \oplus A_2$ can be represented as

$$A_1 \oplus A_2 = \left((D_1 + D_2)^{-1} (D_1 a_1 + D_2 a_2), (D_1 + D_2) \right)_e \tag{28}$$

Proof. In order to obtain $\Pi_{A_1 \oplus A_2}(x)$, we must solve the optimization problem described in the left-hand side of (26). Thus, we have

$$x^* = (D_1 + D_2)^{-1} (D_1 a_1 + D_2 a_2). \tag{29}$$

Substitute x^* into (26), we have

$$k \exp\{-p\} = 1, \tag{30}$$

where

$$p = (D_1 a_1 + D_2 a_2)^t (D_1 + D_2)^{-1} (D_1 a_1 + D_2 a_2) + a_1^t D_1 a_1 + a_2^t D_2 a_2. \tag{31}$$

Thus, we have

$$k = \exp\{p\}, \tag{32}$$

Substituting k into (25) yields (28). □

From (32), the measure of ignorance of $A_1 \oplus A_2$ can be written as

$$I(A_1 \oplus A_2) = a_1^t D_1 a_1 + a_2^t D_2 a_2 - (D_1 a_1 + D_2 a_2)^t (D_1 + D_2)^{-1} (D_1 a_1 + D_2 a_2). \tag{33}$$

In general, the possibility distribution of $A_1 \oplus \cdots \oplus A_n$ can be obtained in the following theorem.

Theorem 4. The combination of *r* possibility distributions A_i, $i = 1, \ldots, r$ can be represented as the following exponential possibility distribution.

$$A_1 \oplus \cdots \oplus A_r = \left(\left(\sum_{i=1}^{r} D_i \right)^{-1} \left(\sum_{i=1}^{r} D_i a_i \right), \sum_{i=1}^{r} D_i \right)_e . \tag{34}$$

The measure of ignorance of $A_1 \oplus \cdots \oplus A_r$ is

$$I(A_1 \oplus \cdots \oplus A_r) = \sum_{i=1}^{r} a_i^t D_i a_i - \left(\sum_{i=1}^{r} D_i a_i \right)^t \left(\sum_{i=1}^{r} D_i \right)^{-1} \left(\sum_{i=1}^{r} D_i a_i \right). \tag{35}$$

It is easy to prove this theorem using the mathematical induction. It follows from Definition 2 that $I(A_1 \oplus \cdots \oplus A_r) \geq 0$. Thus, we have

$$\sum_{i=1}^{r} a_i^t D_i a_i - \left(\sum_{i=1}^{r} D_i a_i \right)^t \left(\sum_{i=1}^{r} D_i \right)^{-1} \left(\sum_{i=1}^{r} D_i a_i \right) \geq 0. \tag{36}$$

It follows from Theorem 4 that

$$A_1 \oplus A_2 \oplus A_3 = A_2 \oplus A_1 \oplus A_3, \tag{37}$$

$$I(A_1 \oplus A_2 \oplus A_3) = I(A_2 \oplus A_1 \oplus A_3), \tag{38}$$

which show that the combination rule is unaffected by any permutation of A_j.

Theorem 5. The measure of fuzziness of $A_1 \oplus A_2$ can be written as

$$H(A_1 \oplus A_2) = \pi^{n/2} \left| (D_1 + D_2)^{-1} \right|^{1/2} \tag{39}$$

and the following inequality holds:

$$H(A_1 \oplus A_2) \leq H(A_i) \quad i = 1, 2. \tag{40}$$

It is clear from Theorems 1~3 that Theorem 5 holds.

Let us consider the following special cases.

1) The case where $A_i = \left(a, D_i^{-1}\right)_e$, $i = 1, \ldots, r$ (the same center vectors)

$$A_1 \oplus \cdots \oplus A_r = \left(a, \sum_{i=1,\ldots,r} D_i\right)_e, \tag{41}$$

$$I(A_1 \oplus \cdots \oplus A_r) = 0, \tag{42}$$

$$H(A_1 \oplus \cdots \oplus A_r) = \pi^{n/2} \left| \left(\sum_{i=1}^{r} D_i \right)^{-1} \right|^{1/2}. \tag{43}$$

2) The case where $A_i = \left(a_i, D^{-1}\right)_e$, $i = 1, \ldots, r$ (the same distribution matrices)

$$A_1 \oplus \cdots \oplus A_r = \left(\sum_{i=1,\ldots,r} a_i / r, rD \right)_e, \tag{44}$$

$$I(A_1 \oplus \cdots \oplus A_r) = \sum_{i=1,\ldots,r} a_i^t D a_i - \left(\sum_{i=1,\ldots,r} a_i \right)^t D \left(\sum_{i=1,\ldots,r} a_i \right) / r, \tag{45}$$

$$H(A_1 \oplus \cdots \oplus A_r) = \pi^{n/2} r^{n/2} \left| D^{-1} \right|^{1/2}. \tag{46}$$

3) The case where $A_i = \left(a, D^{-1}\right)_e$, $i = 1, \ldots, r$ (the same distributions)

$$A_1 \oplus \cdots \oplus A_r = (a, rD)_e, \tag{47}$$

$$I(A_1 \oplus \cdots \oplus A_r) = 0, \tag{48}$$

$$H(A_1 \oplus \cdots \oplus A_r) = \pi^{r/2} r^{-r/2} \left| D^{-1} \right|^{1/2}. \tag{49}$$

In case 3), taking $r \to \infty$, we have $\lim_{r \to \infty} H(A_1 \oplus \cdots \oplus A_r) = 0$. Thus, the combined evidence $A_1 \oplus \cdots \oplus A_r$ supports the center a without ignorance and fuzziness as $r \to \infty$. This is similar to the law of large numbers in probability theory.

4 Possibility Relation Systems

Let a possibility relation R be represented as

$$\Pi_R(x, y) = \exp\left\{ -(x - r_1, y - r_2)^t \begin{bmatrix} D_{11} & D_{12} \\ D_{12}^t & D_{22} \end{bmatrix} \begin{pmatrix} x - r_1 \\ y - r_2 \end{pmatrix} \right\}, \tag{50}$$

where D_{11} and D_{12} are $n \times n$ and $n \times m$ matrices with $n \geq m$, $rank[D_{12}] = m$, $r = [r_1, r_2]^t$ is a center vector and X and Y are input and output spaces, respectively. Setting

$$D_R = \begin{bmatrix} D_{11} & D_{12} \\ D_{12}^t & D_{22} \end{bmatrix}, \tag{51}$$

it is assumed that $D_R > 0$. The parametric representation of (50) is

$$R = (r, D_R)_e . \tag{52}$$

Given a possibilistic input vector $A = (a, D_A)_e$ and a possibility relation $R = (r, D_R)_e$, the possibilistic output $B = (b, D_B)_e$ can be defined by the extension principle as follows.

Definition 5. Given a possibilistic input A and a possibility relation R, the possibility distribution of a possibilistic output vector is defined as

$$\Pi_B(y) = k \max_x \Pi_A(x) \cdot \Pi_R(x, y), \tag{53}$$

where k is a normalized factor such that

$$\max_x \Pi_B(y) = 1. \tag{54}$$

The equation (54) is called a possibility relation system. It follows from (53) and (54) that

$$\max_x \Pi_A(x) \cdot \Pi_R(x, y) = 1/k. \tag{55}$$

Therefore, the measure of ignorance of B can be written as

$$I(B) = -\log(1/k) = \log k . \tag{56}$$

If $I(B) = 0$, A and R are consistent, but if $I(B)$ is large, the obtained possibilistic vector B has lower reliability.

Theorem 6. Given a possibilistic input vector $A = (a, D_A)_e$ and a possibility relation $R = (r, D_R)_e$, the possibilistic output vector B can be written as

$$B = (L^{-1} q, L)_e , \tag{57}$$

where

$$L = D_{22} - D_{12}^t (D_A + D_{11})^{-1} D_{12} , \tag{58}$$

$$q = D_{22} r_2 + D_{12}^t r_1 - D_{12}^t (D_A + D_{11})^{-1} (D_A a + D_{11} r_1 + + D_{12} r_2) . \tag{59}$$

This theorem can be obtained by solving two optimization problems (53) and (54). $L > 0$ can be proved as follows. From $D_A > 0$ and $D_R > 0$, we have

$$D = \begin{bmatrix} D_A + D_{11} & D_{12} \\ D_{12}^t & D_{22} \end{bmatrix} > 0 . \tag{60}$$

Then, setting

$$x = -(D_A + D_{11})^{-1} D_{12}\, y, \tag{61}$$

we have

$$\left(-(D_A + D_{11})^{-1} D_{12}\, y, y\right)^t D \begin{pmatrix} -(D_A + D_{11})^{-1} D_{12}\, y \\ y \end{pmatrix} = y^t \left(D_{22} - D_{12}^t (D_A + D_{11})^{-1} D_{12}\right) y > 0 . \tag{62}$$

The measure of ignorance of B can be represented as

$$I(B) = \log k = a^t D_A a + r^t D_R r - q^t L^{-1} q - (D_A a + D_{11} r_1 + D_{12} r_2)^t (D_A + D_{11})^{-1} (D_A a + D_{11} r_1 + D_{12} r_2). \tag{63}$$

Let us consider three kinds of special cases to clarify the output B and $I(B)$.
1) A crisp input x^0 :

$$\Pi_B = \left(r_2 - D_{22}^{-1} D_{12}^t (x^0 - r_1), D_{22}\right)_e, \tag{64}$$

$$I(B) = (x^0 - r_1)^t \left(D_{11} - D_{12} D_{22}^{-1} D_{12}^t\right)(x^0 - r_1). \tag{65}$$

2) $D_{12} = 0$ (the case where x and y are independent) :

$$\Pi_B = (r_2, D_{22})_e, \tag{66}$$

$$I(B) = a^t D_A a + r_1^t D_{11} r_1 - (D_A a + D_{11} r_1)(D_A + D_{11})^{-1} (D_A a + D_{11} r_1). \tag{67}$$

3) $a = r_1$ (the case where the center vector of possibilistic input is the same as the center vector with respect to x of the possibility relation R) :

$$\Pi_B = (r_2, L)_e, \tag{68}$$

$$I(B) = 0. \tag{69}$$

Let two possibility relations on $X \times Y$ and $Y \times Z$ be denoted as $\Pi_A(x, y)$ and $\Pi_B(y, z)$, respectively, which are possibility distributions on $X \times Y$ and $Y \times Z$. The possibility distribution $\Pi_C(x, z)$ induced from $\Pi_A(x, y)$ and $\Pi_B(y, z)$ is defined as

$$\Pi_C(x, z) = k \max_x \Pi_A(x, y) \cdot \Pi_B(y, z), \tag{70}$$

where k is a normalizing factor such that

$$\max_{x, z} \Pi_C(x, z) = 1. \tag{71}$$

(70) is just an extension of (53). Denoting $\Pi_A(x, y)$ and $\Pi_B(y, z)$ as

$$\Pi_A(x, y) = \exp\left\{-(x - a_x, y - a_y)^t \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^t & A_{22} \end{bmatrix} \begin{pmatrix} x - a_x \\ y - a_y \end{pmatrix}\right., \tag{72}$$

$$\Pi_B(y, z) = \exp\left\{-(y - a_y, z - a_z)^t \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^t & B_{22} \end{bmatrix} \begin{pmatrix} y - a_y \\ z - a_z \end{pmatrix}\right., \tag{73}$$

we can obtain $\Pi_C(x, z)$ by solving two optimization problems (70) and (71) as

$$\Pi_C(x,z)=\exp\left\{-(x-c_x, y-c_y)^t\begin{bmatrix}C_{11} & C_{12}\\ C_{12}^t & C_{22}\end{bmatrix}\begin{pmatrix}x-c_x\\ y-c_z\end{pmatrix}\right\}, \tag{74}$$

where

$$C_{11}=A_{11}-A_{12}(A_{22}+B_{11})^{-1}A_{12}^t, \tag{75}$$

$$C_{22}=B_{22}-B_{12}^t(A_{22}+B_{11})^{-1}B_{12}, \tag{76}$$

$$C_{12}=-A_{12}(A_{22}+B_{22})^{-1}B_{12}, \tag{77}$$

$$c_x=\left(C_{11}-C_{12}C_{22}^{-1}C_{12}^t\right)^{-1}\left(A_{11}a_x+A_{12}a_y-C_{12}C_{22}^{-1}C_{12}^tb_y\right.$$
$$\left.-C_{12}C_{22}^{-1}B_{22}b_z+\left(C_{12}C_{22}^{-1}C_{12}^t-A_{12}\right)(A_{22}+B_{11})^{-1}r_y\right), \tag{78}$$

$$c_z=\left(C_{22}-C_{12}^tC_{22}^{-1}C_{12}\right)^{-1}\left(B_{22}b_x+B_{12}^tA_{12}b_y-C_{12}C_{11}^{-1}A_{11}a_x\right.$$
$$\left.-C_{12}^tC_{11}^{-1}A_{12}a_y+\left(C_{12}^tC_{11}^{-1}A_{12}-B_{12}^t\right)(A_{22}+D_{11})^{-1}r_y\right), \tag{79}$$

$$r_y=-A_{22}a_y+B_{11}b_y+A_{12}^ta_x+B_{12}b_2. \tag{80}$$

The matrix $[C_{ij}]$ in (74) is positive definite because $0\le\Pi_C(x,z)\le 1$ for all x and z due to (70).

5 Conclusion

In this paper, as a representation of evidence, an exponential possibility distribution is considered. The measure of ignorance and fuzziness of an exponential possibility distribution are defined by a normality factor and its area of a possibility distribution, respectively. The definition of the combination of possibility distributions is given. The possibility relational systems are represented as a joint possibility distribution of relationships between inputs and outputs. The rule of the combination of exponential possibility distributions and the exponential possibility of output vector in the possibility relational system are obtained by the optimization problem.

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