論　説

The Effect of an Unobservable Factor on Interest Rates in a Pure Exchange Economy

Hiroshi Morita

1 Introduction

In the framework of standard microeconomics, equilibrium interest rates are decreasing in the level of aggregate consumption. When the level of current consumption becomes high, the marginal rate of substitution decreases and the market-clearing interest rate is determined at a low level. This mechanism is basically the same in the framework of dynamic general equilibrium models. Equilibrium interest rates are decreasing in the level of aggregate consumption while they are increasing in the expected rate of aggregate consumption growth.

While those theories predict negative correlation between interest rates and consumption, it is often observed that short term interest rates tend to be high when the aggregate consumption is high and vice versa. Thus, it seems that the above theories have serious drawbacks to explain the actual movements of interest rates. The objective of this paper is to show that it is not necessarily true and the interest rates can be increasing in the aggregate consumption if we assume that economic variables are partially observable.

In this paper, a pure exchange economy where the aggregate endowment follows a two-factor Gaussian process is explored. We assume that the local drift term in the stochastic differential equation of log endowment follows a Gaussian process, but it is not observable. This means that the level of aggregate endowment is observable, but the expected instantaneous rate of its growth is not. This is a reasonable assumption, because the expected value of growth rate is not directly observable. Under this assumption, it is shown that the estimate of expected instantaneous rate of growth inferred by economic agents can be increasing in the level of endowment. As a result, short term interest rates can be increasing in the level of aggregate consumption in equilibrium.

The effects of an unobservable factor on the market equilibrium have already been studied in a number of papers. Detemple(1986), Dothan and Feldman(1986) explore the economy where an unobservable state variables exist, but these works concentrate on methodological issues. Feldman(1989) and Riedel(2000) focus on the term structure of interest rates, but they examine the functional relationship between estimates inferred by economic agents and interest rates. In this paper, we study the functional relationship between estimates and observable variables in detail to obtain the equilibrium interest rates as a function of observable variables.

This paper is organized as follows. In the next section, we describe a pure exchange economy where agents are identical. In section 3, we investigate how the level of endowment affects the estimate of expected rate of growth. In section 4, we calculate the equilibrium interest rate briefly. In section 5, we impose an assumption on the estimation error process of agents and show that the equilibrium interest rates can be monotone increasing in the level of aggregate endowment. Section 6 states summary and conclusion.

2 Model

Consider a pure exchange economy of a single perishable consumption good. The time span of this economy is $[0,\tau]$. Let (Ω, F, Q) be a complete probability space. The economy is driven by two-dimensional Wiener process $\{Z_t : t \in [0,\tau]\}$ where $Z_t^\top = [Z_{1t}, Z_{2t}]$. We assume that Z_{1t} and Z_{2t} are independent. The economy is endowed with a flow of the consumption good. The rate of aggregate endowment flow is y_t, $t \in [0,\tau]$. In this paper, it is assumed that y_t follows a stochastic differential equation,

$$\frac{dy_t}{y_t} = \mu_t \, dt + \sigma^\top dZ_t, \tag{1}$$

where $\sigma^\top = [\sigma_1, \sigma_2]$ is a vector of constants. Without loss of generality, it is assumed that $\sigma > 0$. The drift term is assumed to follow an Ornstein Uhlenbeck process,

$$d\mu_t = \kappa(\bar{\mu} - \mu_t)\, dt + b^\top dZ_t, \tag{2}$$

where $\bar{\mu}$ and κ are positive constants and $b^\top = [b_1, b_2]$ is a vector of constants[1]. In order to investigate the effect of correlation between changes in y_t and μ_t, we do not restrict the sign of b_1 and b_2.

Throughout this paper, it is assumed that y_t is observable but μ_t is not. It is also assumed that the true value of each parameter is known to all of the agents. Thus, agents infer μ_t, given the past history of level of endowment up to time t. Filtration $\{F_t^y : t \in [0,\tau]\}$ denotes the $Q-$augmentation of natural filtration generated by y_t. It is assumed that the distribution of μ_0 conditioned by F_0^y is normal. This is an important assumption for optimal filtering used in this paper. The estimate of μ_t is denoted as χ_t. By definition, the equation $\chi_t = E[\mu_t \mid F_t^y]$ holds. The estimation error is defined by $\phi_t = E[(\mu_t - \chi_t)^2 \mid F_t^y]$.

Individual agents with identical endowments are assumed to have preferences over the consumption flows,

[1] In discrete time setting, this means that we assume the rate of change in endowment follows ARMA(1,1) process.

$$E\left[\int_0^{\tau} e^{-\delta s} u(c_s)\,ds \middle| F_t^y\right],$$

where felicity function is defined by $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$ for $\gamma > 0$. It is also assumed that the market is frictionless and securities are traded continuously in time. $P(t, T)$ denotes time t price of pure discount bond which promises to pay one unit of consumption good at time $T(T \leq \tau)$.

3 The process of χ_t and ϕ_t

Under the assumption on the conditional distribution of μ_0, the process of estimate, $\{\chi_t : t \geq 0\}$, is known to follow the stochastic differential equation[2],

$$d\chi_t = \kappa(\bar{\mu} - \chi_t)\,dt + \frac{\sigma^\top b + \phi_t}{\sqrt{\sigma^\top \sigma}}\,d\bar{Z}_t, \tag{3}$$

where a one-dimensional Wiener process $\{\bar{Z}_t : t \in [0, \tau]\}$ is defined by,

$$d\bar{Z}_t = \frac{1}{\sqrt{\sigma^\top \sigma}}\left[\frac{dy_t}{y_t} - \chi_t\,dt\right], \tag{4}$$

and the estimation error $\{\phi_t : t \in [0, \tau]\}$ satisfies the following equation,

$$d\phi_t = \left[b^\top b - 2\kappa\phi_t - \frac{(\sigma^\top b + \phi_t)^2}{\sigma^\top \sigma}\right] dt. \tag{5}$$

The initial values, χ_0 and ϕ_0 are defined by,

$$\chi_0 = E[\mu_0 | F_0^y] \tag{6}$$

$$\phi_0 = E\left[(\mu_0 - \chi_0)^2 \middle| F_0^y\right]. \tag{7}$$

The estimation error process (5) implies an ordinary differential equation,

$$\frac{d\phi_t}{dt} = b^\top b - 2\kappa\phi_t - \frac{(\sigma^\top b + \phi_t)^2}{\sigma^\top \sigma}. \tag{8}$$

Suppose that the initial value ϕ_0 is given. Then, the solution of this ordinary differential equation problem is given by,[3]

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[2] For details, see Lipster and Shiryayev(1977).

$$\phi_t = \frac{\bar{\phi} - \underline{\phi}\left(\dfrac{\phi_0 - \bar{\phi}}{\phi_0 - \underline{\phi}}\right)e^{-2\kappa^* t}}{1 - \left(\dfrac{\phi_0 - \bar{\phi}}{\phi_0 - \underline{\phi}}\right)e^{-2\kappa^* t}}, \tag{9}$$

$$\begin{aligned} \text{where } \kappa^* &= \sqrt{\kappa^2 + 2\kappa\frac{\sigma^\top b}{\sigma^\top \sigma} + \frac{b^\top b}{\sigma^\top \sigma}}, \\ \bar{\phi} &= -\sigma^\top b + \sigma^\top \sigma(\kappa^* - \kappa), \\ \underline{\phi} &= -\sigma^\top b + \sigma^\top \sigma(-\kappa^* - \kappa). \end{aligned}$$

Since ϕ_t converges to $\bar{\phi}$ as $t \to \infty$, the parameter $\bar{\phi}$ is interpreted as the stationary level of the estimation error process.

Let us denote the correlation coefficient between changes in y_t and μ_t as ρ. It is easy to prove the following lemma.

Lemma 1 $\kappa^* = 0$ *if and only if the following two equalities hold*,

$$\kappa = \sqrt{\frac{b^\top b}{\sigma^\top \sigma}}, \tag{10}$$

$$\rho = -1. \tag{11}$$

(Proof) From the definition of κ^*, $\kappa^* = 0$ is equivalent to the equality,

$$\kappa^2 \sigma^\top \sigma + 2\kappa\sigma^\top b + b^\top b = 0.$$

Since $\rho = \dfrac{\sigma^\top b}{\sqrt{\sigma^\top \sigma}\sqrt{b^\top b}}$, the left hand side can be reexpressed as,

$$\kappa^2 \sigma^\top \sigma + 2\kappa\sigma^\top b + b^\top b = \kappa^2 \sigma^\top \sigma + 2\rho\kappa\sqrt{\sigma^\top \sigma}\sqrt{b^\top b} + b^\top b.$$

Thus, the inequality,

$$(\kappa\sqrt{\sigma^\top \sigma} - \sqrt{b^\top b})^2 \le \kappa^2 \sigma^\top \sigma + 2\kappa\sigma^\top b + b^\top b \tag{12}$$

holds. Note that (12) holds with equaltiy when $\rho = -1$. Clearly, $\kappa^* = 0$ if and only if the left hand side of (12) is zero and $\rho = -1$. **Q.E.D**

Therefore, strict positiveness of κ^* generically holds. Even in the case of perfect negative correlation, Lebesgue measure in the space $\{(\sigma^\top \sigma, b^\top b)\}$ that those parameters satisfy the condition for $\kappa^* = 0$ is zero. In the sequel, we assume that at least one of (10) and (11) does not hold and κ^* is strictly positive.

[3] This ordinary differential equation belongs to the class of Ricatti equations.

4 Equilibrium prices of pure discount bonds

In our homogeneous economy, the equilibrium prices of pure discount bonds are determined as,

$$\begin{aligned} P(t,T) &= e^{-\delta(T-t)} E\left[\frac{u_c(y_T)}{u_c(y_t)} \middle| F_t^y \right] \\ &= e^{-\delta(T-t)} E\left[\left(\frac{y_T}{y_t} \right)^{-\gamma} \middle| F_t^y \right]. \end{aligned}$$

By the law of iteration, this equality is reexpressed as,

$$\begin{aligned} P(t,T) &= e^{-\delta(T-t)} E\left[E\left[\left(\frac{y_T}{y_t} \right)^{-\gamma} \middle| F_t \right] \middle| F_t^y \right] \\ &= e^{-\delta(T-t)} E[E[exp(-\gamma(ln y_T - ln y_t)) | F_t] | F_t^y]. \end{aligned} \tag{13}$$

Since ln y_T is Gaussian given the information structure $\{F_t\}$, the inner conditional expectation is calculates as,

$$\begin{aligned} &\exp\left(-\gamma \left(\frac{1-e^{-\kappa(T-t)}}{\kappa} \right) (\mu_t - \bar{\mu}) \right) \\ &\times \exp\left(-\gamma \left(\bar{\mu} - \frac{\sigma^\top \sigma}{2} \right) (T-t) + \frac{1}{2}\gamma^2 Var(ln y_T | F_t) \right). \end{aligned}$$

Substituting this into (13) gives,

$$\begin{aligned} P(t,T) &= e^{-\delta(T-t)} E\left[exp\left(-\gamma \left(\frac{1-e^{-\kappa(T-t)}}{\kappa} \right) (\mu_t - \bar{\mu}) \right) \middle| F_t^y \right] \\ &\quad \times \exp\left(-\gamma \left(\bar{\mu} - \frac{\sigma^\top \sigma}{2} \right) (T-t) + \frac{1}{2}\gamma^2 Var(ln y_T | F_t) \right). \end{aligned} \tag{14}$$

Note that the conditional variance $Var(ln y_T | F_t)$ is not random and can be put outside the expectation conditioned by $\{F_t^y\}$.

By Proposition 12.6 in Lipster and Shiryayev(1977), μ_t is Gaussian under $\{F_t^y\}$. Thus, we can express the bond price as,

$$
\begin{aligned}
P(t,T) &= \exp\left[-\delta(T-t) + E\left[-\gamma\left(\frac{1-e^{-\kappa(T-t)}}{\kappa}\right)(\mu_t-\bar{\mu})\middle| F_t^y\right]\right]\\
&\times \exp\left[\frac{1}{2}Var\left[\left(\frac{1-e^{-\kappa(T-t)}}{\kappa}\right)(\mu_t-\bar{\mu})\middle| F_t^y\right]\right]\\
&\times \exp\left(-\gamma\left(\bar{\mu}-\frac{\sigma^\top\sigma}{2}\right)(T-t)+\frac{1}{2}\gamma^2 Var(\ln y_T\,|\,F_t)\right)\\
&= \exp\left[-\delta(T-t) - \gamma\left(\frac{1-e^{-\kappa(T-t)}}{\kappa}\right)(\chi_t-\bar{\mu})\right]\\
&\times \exp\left[\frac{1}{2}\gamma^2\left(\frac{1-e^{-\kappa(T-t)}}{\kappa}\right)^2\phi_t\right]\\
&\times \exp\left(-\gamma\left(\bar{\mu}-\frac{\sigma^\top\sigma}{2}\right)(T-t)+\frac{1}{2}\gamma^2 Var(\ln y_T\,|\,F_t)\right).
\end{aligned}
\tag{15}
$$

In the last equality, we use the fact that $E[\mu_t | F_t^y] = \chi_t$ and $Var[\mu_t | F_t^y] = \phi_t$. By differentiating the negative of log price with respect to T, we obtain the instantaneous forward rate. Denoting instantaneous forward rate with maturity T as $f(t,T)$, we have,

$$
\begin{aligned}
f(t,T) &= \delta + \gamma\left(e^{-\kappa(T-t)}\chi_t + (1-e^{-\kappa(T-t)})\left(\bar{\mu}-\frac{\sigma^\top\sigma}{2}\right)\right)\\
&-\gamma^2\frac{1-e^{-\kappa(T-t)}}{\kappa}e^{-\kappa(T-t)}\phi_t\\
&-\frac{1}{2}\gamma^2\left[\sigma+\left(\frac{1-e^{-\kappa(T-t)}}{\kappa}\right)b\right]^\top\left[\sigma+\left(\frac{1-e^{-\kappa(T-t)}}{\kappa}\right)b\right].
\end{aligned}
\tag{16}
$$

5　Time-homogeneous model

Using the definition of $\overline{Z}_t$, the stochastic differential equation of χ_t is expressed as,

$$
d\chi_t = (\kappa\bar{\mu} - \xi_t\chi_t)\,dt + (\xi_t-\kappa)\left(d\ln y_t + \frac{1}{2}\sigma^\top\sigma dt\right),
\tag{17}
$$

where $\xi_t = \kappa + \dfrac{\sigma^\top b + \phi_t}{\sigma^\top\sigma}$. This leads to the stochastic integral form,

$$\chi_t = \chi_0 e^{-\int_0^t \xi_v dv} + \int_0^t \left(\kappa\bar{\mu} + (\xi_u - \kappa)\frac{\sigma^\top \sigma}{2} \right) e^{-\int_u^t \xi_v dv} du$$
$$+ \int_0^t (\xi_u - \kappa) e^{-\int_u^t \xi_v dv} d\ln y_u .$$

Thus, χ_t depends on the past history of local variation of endowment.

In general, the estimation error deterministically changes over time. But (9) implies ϕ_t converges to $\bar{\phi}$ and the estimation error is approximately equal to $\bar{\phi}$ for sufficiently large t. In this spirit, we impose the following assumption in part for simplicity, and because we want to obtain the functional relationship between y_t and χ_t without ambiguity.

Assumption 1 *The initial estimation error is given by,*

$$\phi_0 = \bar{\phi}. \tag{19}$$

Clearly, $\phi_t = \bar{\phi}$ for all t under this assumption. From the definition of $\bar{\phi}$ and ξ_t, we can easily show the following lemma.

Lemma 2 *Under assumption* 1, $\xi_t = \kappa_*$ *for all* $t \geq 0$.

(proof) As we mentioned, under assumption 1, $\phi_t = \bar{\phi}$ for all $t \geq 0$. Combining this result with the definitions of $\bar{\phi}$ yields $\phi_t = -\sigma^\top b + \sigma^\top \sigma (\kappa_* - \kappa)$ for all $t \geq 0$. Substituting this equation into the definition of ξ_t,

$$\xi_t = \kappa + \frac{\sigma^\top b + \phi_t}{\sigma^\top \sigma},$$

we obtain $\xi_t = \kappa_*$ for all $t \geq 0$. **Q.E.D**

Under assumption 1, χ_t satisfies a time-homogeneous stochastic differential equations. From lemma 2, (18) is reduced to,

$$\chi_t = \chi_0 e^{-\kappa^* t} + \bar{\chi}\left(1 - e^{-\kappa^* t}\right) + (\kappa_* - \kappa) \int_0^t e^{-\kappa^*(t-u)dv} d\ln y_u , \tag{20}$$

where $\bar{\chi} = \left(\frac{\kappa}{\kappa_*}\right)\bar{\mu} + \left(1 - \frac{\kappa}{\kappa_*}\right)\frac{\sigma^\top \sigma}{2}$. Since $\kappa_* \geq 0$, the integral in the right hand side can be interpreted as the weigted average of the past local variations of ln y where heavy weights are put on the recent variations.

By integral by parts, the integral in the right hand side of (20) can be expressed as,

$$\int_0^t e^{-\kappa^*(t-u)} d\ln y_u = \ln y_t - e^{-\kappa^* t} \ln y_0 - \int_0^t \kappa_* e^{-\kappa^*(t-u)} \ln y_u du.$$

Substituting this equation into (20), χ_t can be expressed as,

$$\begin{aligned}\chi_t &= \chi_0 e^{-\kappa^* t} + \bar{\chi}\left(1 - e^{-\kappa^* t}\right) \\ &+ (\kappa^* - \kappa)\left(ln\, y_t - e^{-\kappa^* t} ln\, y_0\right) \\ &- (\kappa^* - \kappa)\left(\int_0^t \kappa^* e^{-\kappa^*(t-u)} ln\, y_u\, du\right).\end{aligned} \tag{21}$$

From this equation, we know that χ_t linearly depends on ln y_t. When y_t increases, does χ_t increase or decrease? The following proposition answers this question.

Proposition 1 *Under the assumption 1, χ_t is increasing in y_t if and only if the following inequality holds,*

$$\rho \geq -\frac{1}{2\kappa}\sqrt{\frac{b^\top b}{\sigma^\top \sigma}}. \tag{22}$$

(proof) From the definition of κ^*, $\kappa^* - \kappa$ is positive if and only if $2\kappa \frac{\sigma^\top b}{\sigma^\top \sigma} + \frac{b^\top b}{\sigma^\top \sigma}$ is positive. This condition is arranged as,

$$-\sigma^\top b \leq \frac{b^\top b}{2\kappa}.$$

Using the definition of ρ, we can change the expression of this inequality to (22). **Q.E.D**

From this result, in the case of $\rho = 0$, changes in ξ_t and y_t are positively correlated. Even if the changes in μ_t and y_t are negatively correlated, changes in χ_t can be positively correlated with the changes in y_t. This is the important effect of unobservability of μ_t on the equilibrium interest rates. To understand this, let us consider the case in which the inequalities,

$$-\frac{b^\top b}{2\kappa} < \sigma^\top b < 0 \tag{23}$$

hold. The second inequality means that changes in μ_t and y_t are negatively correlated. But changes in χ_t is increasing in y_t since the condition in proposition 1 is met by the first inequality. That is, changes in χ_t and y_t are positively correlatted even if changes in μ_t and y_t are negatively correlated. This interesting result holds, because μ_t is unobservable and increase in y_t, for example, makes agents infer that μ_t has become high even under the negative correlation between changes in μ_t and y_t.

As a corollary, we can establish a sufficient condition for positive correlation between changes in χ_t and y_t for any correlation coefficient ρ.

Corollary 1 *Suppose that the following condition is met,*

$$\kappa \leq \frac{1}{2}\sqrt{\frac{\sigma^\top \sigma}{b^\top b}}. \tag{24}$$

Then, under assumption 1, χ_t *is increasing in* y_t *for any correlation coefficient* $\rho \in [-1, 1]$.

(proof) The inequality (24) is equivalent to the inequality $-1 \geq -\frac{1}{2\kappa}\sqrt{\frac{b^\top b}{\sigma^\top \sigma}}$. Combining this inequality and $\rho \geq -1$ yields (22). This concludes the proof. **Q.E.D**

From (16), the instantaneous forward rates are monotone increasing function of χ_t. Thus, we estblish the following proposition.

Proposition 2 *Under assumption 1, the following two statements hold.*
(A)Suppose the condition (22) is met. Then, the instantaneous forward rates in equilibrium are monotone incereasing in the level of endowment.
(B)Suppose the condition (24) is met. Then, the instantaneous forward rates are monotone increasing in the level of endowment for any correlation coefficient $\rho \in [-1, 1]$.

6 Summary and Conclusion

In this paper, we examined how the existence of an unobservable factor affects the market interest rates, assuming that the expected rate of endowment growth is unobservable. This assumption is reasonable, because the expected value of growth rate is generally not observable. The agents in this economy infer the expected rate of endowment growth from the past history of realization value of endowment.

Adding an assumption on the stationarity of estimating error process, we obtain the result: If the correlation between the growth rate and the level of endowment is sufficiently high, then the instantaneous forward rates in equilibrium are monotone increasing in the level of endowment. Even if the correlation coeffeicnt is negative, this property can hold, because the observation that the level of endowment increases, for example, makes agents infer that expected growth rate has become high even under the negative correlation between the expected growth rate and the level of endowment.

Under the assumption on unobservability of expected growth rate, our model is reduced to one-factor term structure model. For empirical studies, at least another risk factor must be introduced. This remains for future researches.

Endnotes

1 In discrete time setting, this means that we assume the rate of change in endowment follows ARMA (1, 1) process.
2 For details, see Lipster and Shiryayev (1977).
3 This ordinary differential equation belongs to the class of Ricatti equations.

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〔Hiroshi Morita, Professor, Faculty of Business Administration, Yokohama National University〕

〔2007年 4 月 6 日受理〕