

On a Class of Discrete Convolution Concave Functions and their Applications¹

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Abstract: We introduce a class of discrete convolution concave functions and show that these new functions share several fundamental properties with continuous concave functions, but also exhibit marked differences as well. These new functions generalize gross substitutes utility functions of Kelso and Crawford (1982), valuated matroids of Dress and Wenzel (1992), $M^\#$ -concave functions of Murota and Shioura (1999), and GM-concave functions of Sun and Yang (2006).

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1 Introduction

Convexity plays a central role in the study of optimization problems in engineering, management and economics. Ordinary convex functions have been shown to exhibit many interesting properties, see e.g., Rockafellar (1970). Unfortunately, a simple analogy of convex function does not work any more when the variables take only integer values. In the literature, a great deal of research has been done to identify discrete functions which have nice properties as enjoyed by ordinary convex functions. For instance, Frank (1982) established the first discrete separation theorem for submodular set functions. Kelso and Crawford (1982) introduced the gross substitutes condition for the utility functions in an exchange economy with indivisible goods and proved the existence of a Walrasian equilibrium under the condition. Lovász (1983) demonstrated a close relationship between submodularity and convexity. Camerini et al.

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(1989) introduced the class of quasi-separable convex functions and proposed a greedy algorithm for those functions. Favati and Tardella (1990) introduced the class of discrete midpoint convex functions and proved that local minimality leads to global minimality for those functions. Dress and Wenzel (1992) formulated the concept of valuated matroid and investigated its various properties. Murota (1996) generalized the valuated matroid to the lattice of integers and showed several interesting properties of this generalization. Murota and Shioura (1999) extended this generalization further and called this extension an $M^\#$ -concave function. Sun and Yang (2004) introduced the class of convolution concave functions further generalizing $M^\#$ -concave functions and illustrated the application of this class of utility functions in equilibrium models with indivisibilities. Extending $M^\#$ -concave functions, Sun and Yang (2006) proposed the class of GM-concave functions as new examples of convolution concave functions and generalized the well-known gross substitutes condition to the gross substitutes and complements condition and proved the existence of a Walrasian equilibrium under the condition.

The purpose of this paper is to introduce a number of natural classes of convolution concave functions. In particular, we show that these new classes of convolution concave functions preserve several fundamental properties of valuated matroids and $M^\#$ -concave functions such as (1) local optimality implies global optimality; (2) discrete separation; (3) the convolution concavity property. While our generalizations and valuated matroids, $M^\#$ -concave functions have several properties in common, there is also a marked difference in that any $M^\#$ -concave function must be submodular but our generalizations need not be submodular nor supermodular. In economic terms, if a consumer has an $M^\#$ -concave function as her utility over n types of indivisible goods, then the consumer actually regards all the goods as substitutes in the sense of Kelso and Crawford (1982) (see Fujishige and Yang (2003), Murota (2003)). Substitutes exclude complements. In many important economic environments, complements and substitutes are often jointly observed, such as workers and machines, tables and chairs, computers and software packages. The existing theory of discrete convex analysis, however, cannot handle such complex and also realistic situations. Our generalizations are so general and so interesting that they can cope with the mentioned complex situations and also include the typical pattern of substitutability and complementarity among goods as studied by Sun and Yang (2006) (see also Milgrom (2007)), as special cases.

2 Main Results

Let I_k denote the set of the first k positive integers. The set \mathbb{R}^n denotes the n -dimensional Euclidean space and \mathbb{Z}^n the set of all integer lattice points in \mathbb{R}^n . The vector $\mathbf{0}$ denotes the vector of zeros. The vector $e(i)$, $i \in I_n$, is the i th unit vector of \mathbb{R}^n . Furthermore, $x \cdot y$ means the inner product of vectors x and y . Given $x, y \in \mathbb{R}^n$, define

$$x \vee y = (\max\{x_1, y_1\}, \dots, \max\{x_n, y_n\}), \quad x \wedge y = (\min\{x_1, y_1\}, \dots, \min\{x_n, y_n\}).$$

Given a function $f: \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{\pm\infty\}$, define the effective domain of f by

$$\text{dom}(f) = \{x \in \mathbb{Z}^n \mid f(x) \text{ is finite}\}.$$

We assume that the effective domain, $\text{dom}(f)$, is not empty for any function f under consideration.

A function $f: \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ is said to be *discrete concave* if, for all points x^1, x^2, \dots, x^l in \mathbb{Z}^n with all convex parameters $\lambda_1, \lambda_2, \dots, \lambda_l$ satisfying $\sum_{h=1}^l \lambda_h x^h \in \mathbb{Z}^n$, it holds

$$f\left(\sum_{h=1}^l \lambda_h x^h\right) \geq \sum_{h=1}^l \lambda_h f(x^h).$$

A function $f: \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be *discrete convex* if $-f$ is discrete concave. Note that if a function $f: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ is a concave function, its restriction on the integer lattice \mathbb{Z}^n is obviously discrete concave.

A function $f: \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be *submodular* if

$$f(x) + f(y) \geq f(x \wedge y) + f(x \vee y), \forall x, y \in \mathbb{Z}^n.$$

A function $f: \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ is said to be *supermodular* if $-f$ is submodular.

Let f_1 and f_2 be functions mapping from \mathbb{Z}^n to $\mathbb{R} \cup \{-\infty\}$. Define $f_1 \oplus f_2: \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ by

$$f_1 \oplus f_2(x) = \sup \{f_1(x^1) + f_2(x^2) \mid x^1 + x^2 = x, x^1, x^2 \in \mathbb{Z}^n\},$$

provided that the supremum is not equal to $+\infty$.

Definition 2.1 A class $F = \{f \mid f: \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{-\infty\}\}$ of functions is said to be *convolution concave* if the following conditions are satisfied:

- (i) For every $f \in F$, f is discrete concave;
- (ii) For every f and g in F , we also have $f \oplus g$ in F .

Definition (i) is fairly standard and general. Definition (ii) states that the class of functions under consideration is closed under the supremum convolution operation. If F is a class of convolution concave functions, then $g = \{-f \mid f \in F\}$ is called a *class of convolution convex functions*. The notion of convolution concavity was introduced in Sun and Yang (2004) and motivated from a study on the existence of a Walrasian equilibrium.

One can analogously define the above concepts for the continuous case. In this case, we have the following simple lemma; see Rockafellar (1970).

Lemma 2.2 If $f_i: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$, $i=1, 2$, are concave, then the function $f_1 \oplus f_2$ is also concave provided $f_1 \oplus f_2$ does not take the value $+\infty$.

Let $F = \{f \mid f: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\} \text{ is concave and bounded from above}\}$. Then, it follows from Lemma 2.2 that F is convolution concave. Thus, in the continuous case, the class of all concave functions is essentially convolution concave. Unfortunately, in the discrete case, even if $f_i: \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{-\infty\}$, are concave, their convolution function $f_1 \oplus f_2$ need not be concave. This is a crucial difference between continuous concave functions and discrete concave functions. To preserve the convolution concavity in the discrete case, one needs to take combinatorial or discrete nature of discrete functions into account. In

the following we will introduce a number of natural classes of convolution concave functions.

Given $x \in \mathbb{R}^n$, define $\text{supp}^+(x) = \{k \in I_n \mid x_k > 0\}$ and $\text{supp}^-(x) = \{k \in I_n \mid x_k < 0\}$. An $n \times n$ matrix U is said to be *unimodular* if U is integral and has determinant ± 1 . Note that U^{-1} is also unimodular if U is unimodular. The identity matrix I is the simplest unimodular matrix. Let \mathcal{U} denote the family of all $n \times n$ unimodular matrices.

Definition 2.3 Given an $n \times n$ unimodular matrix U , a function $f: \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ is said to be a *GM(U)-concave function* if for $x, y \in \text{dom}(f)$ and $k \in \text{supp}^+(U(x-y))$, it satisfies

$$\begin{aligned} f(x) + f(y) & \leq \max \left[f(x - U^{-1}e(k)) + f(y + U^{-1}e(k)), \right. \\ & \left. \max_{l \in \text{supp}^-(U(x-y))} \left\{ f(x - U^{-1}(e(k) - e(l))) + f(y + U^{-1}(e(k) - e(l))) \right\} \right]. \end{aligned}$$

Note 1: A GM(U)-concave function coincides with an $M^\#$ -concave function of Murota and Shioura (1999) if U is the identity matrix. An $M^\#$ -concave function is a generalization of an *M-concave* function of Murota (1996), which in turn is a generalization of a valuated matroid of Dress and Wenzel (1992). Clearly, the class of GM(U)-concave functions is substantially and strictly larger than that of $M^\#$ -concave functions. As shown in Fujishige and Yang (2003), Murota (2003), there is an equivalence between $M^\#$ -concavity and the well-known gross substitutes of Kelso and Crawford (1982). This means that if a consumer has an $M^\#$ -concave utility function over n types of indivisible goods, then she views all goods as substitutes. A simple example is tables which have the same function for the consumer but may be different in quality.

Note 2: Let (S_1, S_2) be a partition of the set I_n . Let a unimodular matrix U be given by $U = [e(i), i \in S_1; -e(i), i \in S_2]$. With respect to this unimodular matrix U , a GM(U)-concave function coincides with a GM concave function of Sun and Yang (2006). In this case, as shown in Sun and Yang (2006), if a consumer has a GM concave utility function over n types of indivisible goods, then the consumer regards the goods of types in each set S_i as substitutes but the goods across S_1 and S_2 as complements. A simple example is tables and chairs. The consumer sees tables (chairs) as substitutes but tables and chairs as complements. Even in this relatively simple case, the GM(U)-concave functions capture the essence of some fundamental economic phenomena beyond what the class of $M^\#$ -concave functions can deal with.

It is known from Murota (2003) and Fujishige (2005) that every $M^\#$ -concave function must be submodular. The following lemma shows, however, that a GM(U)-concave function need not be submodular nor supermodular.

Lemma 2.4 For any $U \in \mathcal{U}$ not being the identity matrix, a GM(U)-concave function is not necessarily submodular nor supermodular and thus need not be $M^\#$ -concave.

Proof: Consider the function $g: \mathbb{R} \rightarrow \mathbb{R}$ given by $g(x) = -|x|$. Clearly, g is concave. Then define $f: \mathbb{Z}^3 \rightarrow \mathbb{R}$ by $f(x) = -|x_1 - x_2 + x_3|$. It can be shown that f is GM(U)-concave with $U = [e(1), -e(2), e(3)]$. First consider the points $x^1 = (1, 0, 0)$ and $x^2 = (0, 1, 0)$. Then we have

$$f(x^1) + f(x^2) = -2 < f(x^1 \vee x^2) + f(x^1 \wedge x^2) = f(1, 1, 0) + f(0, 0, 0) = 0.$$

Thus, f is not submodular and so f is not $M^\#$ -concave.

Next consider the points $y^1 = (1, 0, -1)$ and $y^2 = (-1, 0, 1)$. Then we have

$$f(y^1) + f(y^2) = 0 > f(y^1 \vee y^2) + f(y^1 \wedge y^2) = f(1, 0, 1) + f(-1, 0, -1) = -4.$$

Thus, f is not supermodular. □

The next result shows that $GM(U)$ -concave functions and $M^\#$ -concave functions are unimodularly convertible, and also gives a way of constructing a $GM(U)$ -concave function from any given $M^\#$ -concave function and any given unimodular matrix.

Lemma 2.5 (1) For a given $U \in U$, let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ be a $GM(U)$ -concave function. Then the function $F : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ defined by $F(x) = f(U^{-1}x)$ is $M^\#$ -concave.

(2) Let $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ be an $M^\#$ -concave function.

Then, for any $U \in U$ the function $F : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ defined by $F(x) = f(Ux)$ is $GM(U)$ -concave.

Proof: (1) Let us first note the fact that the inverse matrix of a unimodular matrix is also a unimodular matrix. For any pair x and $y \in \text{dom}(F)$, let $x' = U^{-1}x, y' = U^{-1}y$, and $k \in \text{supp}^+(x - y)$. Then we have $x = Ux', y = Uy'$, and $k \in \text{supp}^+(U(x' - y'))$. Thus, we obtain that

$$\begin{aligned} F(x) + F(y) &= f(U^{-1}x) + f(U^{-1}y) = f(x') + f(y') \\ &\leq \max \left[f(x' - U^{-1}e(k)) + f(y' + U^{-1}e(k)), \right. \\ &\quad \left. \max_{l \in \text{supp}^-(U(x' - y'))} \left\{ f(x' - U^{-1}(e(k) - e(l))) + f(y' + U^{-1}(e(k) - e(l))) \right\} \right] \\ &= \max \left[f(U^{-1}(x - e(k))) + f(U^{-1}(y - e(k))), \right. \\ &\quad \left. \max_{l \in \text{supp}^-(U(x' - y'))} \left\{ f(U^{-1}(x - e(k) + e(l))) + f(U^{-1}(y - e(k) - e(l))) \right\} \right] \\ &= \max \left[F(x - e(k)) + F(y + e(k)), \right. \\ &\quad \left. \max_{l \in \text{supp}^-(x - y)} \left\{ F(x - e(k) + e(l)) + F(y + e(k) - e(l)) \right\} \right]. \end{aligned}$$

Therefore, the function $F(x) = f(U^{-1}x)$ is $M^\#$ -concave.

In a similar way we can prove Part (2). □

The following result gives yet another way of constructing $GM(U)$ -concave functions.

Lemma 2.6 Let $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$ be any concave function and U be any $n \times n$ unimodular matrix. Then the function $F : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ given by $F(x) = f\left(\sum_{k=1}^n U_k \cdot x\right)$ is $GM(U)$ -concave, where U_k is the k th row of U .

Now we will show that $GM(U)$ -concave functions possess the following three fundamental and interesting properties as $M^\#$ -concave functions have. But the submodularity of $M^\#$ -concave functions cannot be preserved by $GM(U)$ -concave functions, as has been shown in Lemma 2.4.

Theorem 2.7

- (1) If $f_i : \mathbf{Z}^n \rightarrow \mathbb{R} \cup \{-\infty\}$, $i=1, 2$, are GM(U)-concave functions, then the function $f_1 \oplus f_2$ is also GM(U)-concave provided $f_1 \oplus f_2$ does not take the value $+\infty$.
- (2) If $f : \mathbf{Z}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ is GM(U)-concave, f is discrete concave.
- (3) Let $f : \mathbf{Z}^n \rightarrow \mathbb{Z} \cup \{-\infty\}$ be a GM(U)-concave function and $g : \mathbf{Z}^n \rightarrow \mathbb{Z} \cup \{+\infty\}$ be a GM(U)-convex function such that $\text{dom}(f) \cap \text{dom}(g) \neq \emptyset$. If $f(x) \leq g(x)$ for every $x \in \mathbf{Z}^n$, there exist $\alpha^* \in \mathbb{Z}$ and $p^* \in \mathbf{Z}^n$ such that

$$f(x) \leq \alpha^* + p^* \cdot x \leq g(x)$$

for every $x \in \mathbf{Z}^n$.

Proof: We give a proof for Part (1). Let $F_i(x) = f_i(U^{-1}x)$ for $i=1, 2$. Then by Lemma 2.5 we see that each F_i is $M^\#$ -concave as a function defined $\mathbf{Z}^n \rightarrow \mathbb{R} \cup \{-\infty\}$. Note that

$$\begin{aligned} F_1 \oplus F_2(x) &= \sup \{F_1(x^1) + F_2(x^2) \mid x^1 + x^2 = x, x^1, x^2 \in \mathbf{Z}^n\} \\ &= \sup \{f_1(U^{-1}x^1) + f_2(U^{-1}x^2) \mid x^1 + x^2 = x, x^1, x^2 \in \mathbf{Z}^n\} \\ &= \sup \{f_1(U^{-1}x^1) + f_2(U^{-1}x^2) \mid U^{-1}x^1 + U^{-1}x^2 = U^{-1}x, U^{-1}x^1, U^{-1}x^2 \in \mathbf{Z}^n\} \\ &= \sup \{f_1(y^1) + f_2(y^2) \mid y^1 + y^2 = U^{-1}x, y^1, y^2 \in \mathbf{Z}^n\} \\ &= f_1 \oplus f_2(U^{-1}x). \end{aligned}$$

Conversely, we also have

$$f_1 \oplus f_2(x) = F_1 \oplus F_2(Ux).$$

Note that $f_1 \oplus f_2 < +\infty$ implies $F_1 \oplus F_2 < +\infty$. Now it follows from Theorem 6.15 (1) of Murota (2003) that $F_1 \oplus F_2$ is also $M^\#$ -concave. Finally, it follows from $f_1 \oplus f_2(x) = F_1 \oplus F_2(Ux)$ and Lemma 2.5 that $f_1 \oplus f_2$ is also GM(U)-concave.

Parts (2) and (3) can be proved similarly. Precisely, Part (2) follows from Lemma 2.5 and Theorem 6.42 of Murota (2003). Part (3) follows from Lemma 2.5 and Theorem 8.15 of Murota (2003). \square

Now it follows from Theorem 2.7 (1) and (2) that for any given $U \in U$,

$$F = \{f \mid f : \mathbf{Z}^n \rightarrow \mathbb{R} \cup \{-\infty\} \text{ is GM(U)-concave and bounded from above} \}$$

is a class of convolution concave functions.

We also see that the global optimality of a GM(U)-concave function is guaranteed by the local optimality.

Theorem 2.8 For a GM(U)-concave function f and a point $x \in \text{dom}(f)$,

$$f(x) \geq f(y) \quad (\forall y \in \mathbf{Z}^n) \iff \begin{cases} f(x) \geq f(x + U^{-1}(e(k) - e(l))) & (\forall k, l \in N), \\ f(x) \geq f(x \pm U^{-1}e(k)) & (\forall k \in N). \end{cases}$$

Proof: Let $F(y) = f(U^{-1}y)$. Then by Lemma 2.5 we see that F is $\mathbf{M}^\#$ -concave as a function mapping from \mathbf{Z}^n to $\mathbb{R} \cup \{-\infty\}$. And we have $f(y) = F(Uy)$. Therefore, it follows from Theorem 6.26 of Murota (2003) that

$$F(Ux) \geq F(y) \quad (\forall y \in \mathbf{Z}^n) \iff \begin{cases} F(Ux) \geq F(Ux + e(k) - e(l)) & (\forall k, l \in N), \\ F(Ux) \geq F(Ux \pm e(k)) & (\forall k \in N). \end{cases}$$

That is,

$$\begin{aligned} f(x) \geq f(y) \quad (\forall y \in \mathbf{Z}^n) &\iff f(x) \geq f(U^{-1}y) \quad (\forall k \in \mathbf{Z}^n) \\ &\iff \begin{cases} f(x) \geq f(x + U^{-1}(e(k) - e(l))) & (\forall k, l \in N), \\ f(x) \geq f(x \pm U^{-1}e(k)) & (\forall k \in N). \end{cases} \end{aligned}$$

□

We wrap up this paper with an economically meaningful example. Following Shapley (1962), we consider an optimal assignment problem. A firm has a set S of workers and a set S^c of machines. Let $N = S \cup S^c$. Let $r(i, j) \geq 0$ denote the revenue resulting from the assignment of worker $i \in S$ to machine $j \in S^c$. In general, when the firm uses a set A of workers and machines, the revenue $u(A)$ of these workers and machines to the firm is determined by the pairwise combinations of worker and machine that the members in A can generate, and is given by

$$u(A) = \max \{r(j_1, k_1) + r(j_2, k_2) + \dots + r(j_l, k_l)\}$$

with the maximum to be taken over all sets $\{(j_1, k_1), (j_2, k_2), \dots, (j_l, k_l)\}$ of l distinct worker-machine pairs in A . In this way, we have defined the utility function $u: 2^N \rightarrow \mathbb{R}$. Shapley proves that this utility function u is supermodular w.r.t. any worker-machine pair but is submodular w.r.t. any machine-machine or worker-worker pair. Sun and Yang (2006) have sharpened his result as follows. As an example, this also illustrates the broad applicability of our new generalizations.

Theorem 2.9 The defined utility function u is a GM concave function.

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