論　説

Surjective Function Theorems with Economic Application

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Abstract: Given the unit sphere S^n, we prove the following theorem and several extensions: For any continuous function $f: S^n \mapsto S^n$, if f has no fixed point or if f has no antifixed point in S^n, then f is surjective and has a point $x^* \in S^n$ such that $f(x^*) = -f(-x^*)$. We also apply this result to a competitive exchange economy and demonstrate the existence of an equilibrium in the economy.

Keywords: Surjective function, fixed point, antifixed point, Walrasian equilibrium, satiation.

JEL classification: C78, D71

1 Main Results

In this paper we present several surjective function theorems which are not only interesting on their own but also fundamental and have important economic applications as well. Our approach is a topological one. But first let us review some basic concepts. A function $f: D \mapsto I$ is said to be *surjective* (or f is said to map D *onto* I) if every element of I is the image of some element of D under the function f, i.e., $f(D) = I$. f is said to have *a fixed point* (*an antipodal point*, *an antifixed poin*) in D if there exists $x \in D$ such that $f(x) = x$ ($f(x) = -f(-x), f(x) = -x$). Let $n \geq 2$ denote any integer number, $\mathbb{R}^n$ the n-dimensional Euclidean space, and $x \cdot y = \sum_i x_i y_i$ the inner product of vectors x and y. We write $x \in \mathbb{R}^{n+1}$ by $x = (x_0, x_1, \cdots, x_n)$ or $x = (x_0, x_1, \cdots, x_n)^\top$. Furthermore, define $B^{n+1} = \{x \in \mathbb{R}^{n+1} \mid x \cdot x \leq 1\}$ (i.e. the $(n+1)$-dimensional unit ball), $B^n = \{x \in B^{n+1} \mid x_0 = 0\}$, $S^n = \{x \in \mathbb{R}^{n+1} \mid x \cdot x = 1\}$ (i.e. the n-dimensional unit sphere), $S^{n-1} = \{x \in S^n \mid x_0 = 0\}$, $e_0 = (1, 0, \cdots, 0) \in \mathbb{R}^{n+1}$, and $\mathbf{0} = (0, \cdots, 0) \in \mathbb{R}^{n+1}$.

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Theorem 1.1 *For any continuous function* $f: S^n \mapsto S^n$, *if* f *has no fixed point in* S^n, *then* f *must be surjective. Furthermore, there exists* $x^* \in S^n$ *such that* $f(x^*) = -f(-x^*)$.

Proof: Suppose to the contrary that f is not surjective. Then there would exist some $y^* \in S^n$ which is not in $f(S^n)$. Without loss of generality, we may assume that $y^* = e_0$. Thus, it follows from the continuity of f on the compact set S^n that there exists a positive δ such that $f_0(x) \leq 1 - \delta$ for all $x \in S^n$. Let $C = \{x \in S^n \mid x_0 \leq 1 - \delta\}$ and $C^b = \{x \in S^n \mid x_0 = 1 - \delta\}$. We have that $f(S^n) \subset C$. Define the function $g: C \mapsto B^n$ by

$$g(x) = \frac{1}{2-\delta}\sqrt{\frac{1+x_0}{1-x_0}}\,(0, x_1, \cdots, x_n)^\top.$$

Its inverse $h = g^{-1}: B^n \mapsto C$ is given by for $x \neq 0$

$$h_i(x) = \begin{cases} (2-\delta)\sqrt{x \cdot x} - 1, & \textit{if} \quad i = 0; \\ (2-\delta)\,x_i\sqrt{\dfrac{2-(2-\delta)\sqrt{x \cdot x}}{(2-\delta)\sqrt{x \cdot x}}}, & \textit{if} \quad i = 1, \cdots, n, \end{cases}$$

with $h(0) = -e_0$. It is easy to see that $h(x)$ converges to $-e_0$ as x goes to 0. Thus, both g and h are continuous functions. It may be difficult to figure out how g is constructed. Geometrically, we can visualize the idea in the sphere S^2 being imagined as the surface of the earth. Given any point P on the arctic circle C^b, there is a unique longitude line passing through P which links both the north pole e_0 and the south pole $-e_0$. This longitude intersects the equator line S^{n-1} uniquely at one point, say Q. Clearly, the section of the longitude line between P and the south pole is homeomorphic to the straight line between the core $\mathbf{0}$ and Q. Function g maps P to Q and the south pole to the core.

Consequently, we obtain a continuous function $g \circ f \circ h$ mapping from the convex and compact set B^n into itself. By using Brouwer's fixed point theorem, we know there exists $z^* \in B^n$ such that $z^* = g \circ f \circ h(z^*)$. Setting $x^* = h(z^*)$, we obtain $x^* = f(x^*) \in C$. This contradicts the hypothesis that f has no fixed point in S^n.

Now we prove the last part. Since f has no fixed point, it follows from Corollary 4 (c) of Whittlesey (1963) or Milnor (1965) that f must have an antipodal point. We complete the proof. □

In term of equation theory, the above theorem states that for each $y \in S^n$, the equation $f(x) = y$ has a solution. But we still do not know how to compute such solutions, although there exist several methods for computing fixed points (see Scarf (1973) and Yang (1999)).

Corollary 1.2 *For any continuous function* $f: S^n \mapsto S^n$, *if* $x \cdot f(x) \leq 0$ *for every* $x \in S^n$, *then* f *must be surjective.*

Proof: Suppose to the contrary that f has a fixed point $x \in S^n$. Then $0 < x \cdot x = x \cdot f(x) \leq 0$ which is impossible. □

The above corollary is not so simple as it might appear. In fact we will soon show that it is at least as powerful as the Brouwer fixed point theorem. We prove Brouwer theorem via Corollary 1.2: If $f: B^n \mapsto B^n$ is a continuous function, then there exists $x \in B^n$ such that $f(x^*) = x^*$. Suppose to the contrary that Brouwer theorem is false. Then it holds $f(x) - x \neq 0$ for every $x \in B^n$. Let $g(x) = f(x) - x$. Then

$$x \cdot g(x) = x \cdot (f(x) - x) = x \cdot f(x) - x \cdot x = x \cdot f(x) - 1 < 0$$

for all $x \in B^n \cap S^n = S^{n-1}$. We define the function $h: S^n \mapsto S^n$ as follows: For $x \in S^n$, let $y(x) = (0, x_1, \cdots, x_n)^\top$. Obviously, $y(x) \in B^n$. If $y(x) \cdot g(y(x)) \leq 0$, define

$$h(x) = \frac{g(y(x))}{\sqrt{g(y(x)) \cdot g(y(x))}}.$$

If $y(x) \cdot g(y(x)) > 0$, it is easy to see that $\sum_{i=1}^{n} x_i^2 < 1$ and $x_0 \neq 0$. Define

$$z_0 = -\frac{y(x) \cdot g(y(x))}{x_0}, \quad z = (z_0, 0, \cdots, 0)^\top \in \mathbb{R}^{n+1},$$

and

$$h(x) = \frac{g(y(x)) + z}{\sqrt{(g(y(x)) + z) \cdot (g(y(x)) + z)}}.$$

Since $y(x) \cdot g(y(x))$ is a continuous function in x on a compact set, then there exists some $\delta > 0$ such that

$$y(x) \cdot g(y(x)) \leq -\delta$$

for all $x \in S^{n-1}$. Furthermore, there exists a positive ϵ such that $y(x) \cdot g(y(x)) \geq 0$ implies $|x_0| \geq \epsilon$. Now it is readily verified that h is a continuous function and $x \cdot h(x) \leq 0$ for all $x \in S^n$. So all conditions of Corollary 1.2 are met. Then h must be surjective. But it is impossible since there does not exist any $x \in S^n$ such that $h(x) = e_0$. □

Corollary 1.3 *For any continuous function $f: S^n \mapsto S^n$, if f has no antifixed point in S^n, then f must be surjective. Furthermore, there exists $x^* \in S^n$ such that $f(x^*) = -f(-x^*)$.*

Proof: Let $g(x) = -f(x)$ for all $x \in S^n$. Clearly, g is continuous on S^n, $g(S^n) \subset S^n$, and g has no fixed point in S^n. It follows from Theorem 1.1 that g is surjective and hence so is f. □

Corollary 1.4 *For any continuous function $f: S^n \mapsto S^n$, if $x \cdot f(x) \geq 0$ for every $x \in S^n$, then f*

must be surjective.

Proof: Suppose to the contrary that f has an antifixed point $y \in S^n$. Then $0 > -y \cdot y = f(y) \cdot y \geq 0$ which is impossible. By Corollary 1.3 f is surjective. □

The next result is due to Whittlesey (1963). Here we give an alternative proof which is conceptually much simpler than Whittlesey's.

Theorem 1.5 *For any continuous function $f: S^n \mapsto S^n$, if $f(x) = -f(-x)$ for every $x \in S^n$, then f must be surjective.*

Proof: Suppose to the contrary that f is not surjective. Then there would exist some $y^* \in S^n$ which is not in $f(S^n)$. Without loss of generality, we may assume that $y^* = e_0$.
By the assumption we know that there does not exist any $x \in S^n$ such that $f(x) = -e_0$, either. Now we define the function $g: S^n \mapsto B^n$ by

$$g_i(x) = \begin{cases} 0, & \text{if } \; i = 0; \\ \dfrac{f_i(x)}{\sqrt{\sum_{j=1}^{n} f_j^2(x)}}, & \text{if } \; i = 1, \cdots, n. \end{cases}$$

It follows from the assumption that $g(x) = -g(-x)$ for every $x \in S^n$. Clearly, g is a continuous function. This is impossible according to Borsuk-Ulam theorem (see Yang (1999)) which says that if $l: S^n \mapsto B^n$ is a continuous function, then there exists $x \in S^n$ such that $l(x) = l(-x)$. □

It will be shown that Theorem 1.5 is actually equivalent to the Borsuk-Ulam theorem (see Yang (1999)) which says that if $f: S^n \mapsto B^n$ is a continuous function, then there exists $x \in S^n$ such that $f(x) = f(-x)$. Suppose that the latter theorem is false. Let $g(x) = f(x) - f(-x)$ for all $x \in S^n$. Then $g(x) \neq 0$ and $g(x) = -g(-x)$ for all $x \in S^n$. Define the function $h: S^n \mapsto S^n$ by

$$h(x) = \frac{g(x)}{\sqrt{g(x) \cdot g(x)}}.$$

Clearly, all conditions of Theorem 1.5 are satisfied. Then h must be surjective. But it is impossible since $h_0(x) = 0$ for all $x \in S^n$.

We point out that our first surjective theorem can be extended to the point-to-set mappings as follows.

Theorem 1.6 *For any upper semi-continous point-to-set mapping $F: S^n \Longrightarrow \mathbb{R}^{n+1}$, if there are no $x \in S^n$ and $\alpha > 0$ such that $0 \in f(x)$ or $\alpha x \in F(x)$, then F must be surjective in the sense* *that for any* $v \in S^n$ *there are* $x \in S^n$ *and* $\alpha > 0$ *such that* $\alpha v \in F(x)$.

In the remaining section we will prove two more results. Define

$$S_+^n = \{x \in S^n \mid x_0 > 0\} \text{ and } S_-^n = \{x \in S^n \mid x_0 < 0\}.$$

Theorem 1.7 *For any continuous function* $f: S^n \mapsto S^n$, *if it satisfies*

$$f_0(0, x_1, \cdots, x_n) = 0$$
$$f_i(0, x_1, \cdots, x_n) = -f_i(0, -x_1, \cdots, -x_n), i = 1, \cdots, n$$

for all $(0, x_1, \cdots, x_n) \in S^{n-1}$, *then either* $S_+^n \subset f(S^n)$ *or* $S_-^n \subset f(S^n)$ *or both.*

Proof: Suppose to the contrary that there exist two points $\bar{y} = (\bar{y}_0, \bar{y}_1, \cdots, \bar{y}_n) \in S_+^n$ and $\tilde{y} = (\tilde{y}_0, \tilde{y}_1, \cdots, \tilde{y}_n) \in S_-^n$ neither of which belongs to $f(S^n)$. We define $h: B^n \mapsto S^n$ by

$$h(0, x_1, \cdots, x_n) = (-\sqrt{1 - x \cdot x}, x_1, \cdots, x_n);$$

$h^{-1}: S^n \mapsto B^n$ *by*

$$h^{-1}(x_0, x_1, \cdots, x_n) = (0, x_1, \cdots, x_n);$$

$\bar{g}: B^n \setminus \{h^{-1}(\bar{y})\} \mapsto S^{n-1}$ with $\bar{g}(x)$ equal to the intersection point of the straight line going through x and $h^{-1}(\bar{y})$ on S^{n-1};
$\hat{g}: B^n \setminus \{h^{-1}(\tilde{y})\} \mapsto S^{n-1}$ with $\hat{g}(x)$ equal to the intersection point of the straight line going through x and $h^{-1}(\tilde{y})$ on S^{n-1}.

It is easy to verify that h, h^{-1}, $\bar{g}$, and $\hat{g}$ are continuous functions. Now we construct the function $F: B^n \mapsto S^{n-1}$ by

$$F(0, x_1, \cdots, x_n) = \begin{cases} \bar{g} \circ h^{-1} \circ f \circ h(0, x_1, \cdots, x_n), & \text{if} \quad f_0 \circ h(0, x_1, \cdots, x_n) > 0, \\ \hat{g} \circ h^{-1} \circ f \circ h(0, x_1, \cdots, x_n), & \text{if} \quad f_0 \circ h(0, x_1, \cdots, x_n) < 0, \\ f \circ h(0, x_1, \cdots, x_n), & \text{if} \quad f_0 \circ h(0, x_1, \cdots, x_n) = 0. \end{cases}$$

Note that $h^{-1}(x) = \bar{g}(x) = \hat{g}(x) = x$ for all $x \in S^{n-1}$. Then for any $x \in B^n$ with $f_0 \circ h(x) = 0$, it holds that

$$\bar{g} \circ h^{-1} \circ f \circ h(x) = \hat{g} \circ h^{-1} \circ f \circ h(x) = f \circ h(x).$$

From the continuity of h, h^{-1}, $\bar{g}$, and $\hat{g}$, F is a continuous function and furthermore it satisfies that

$$F(0,x_1,\cdots,x_n)=f\circ h(0,x_1,\cdots,x_n)=f(0,x_1,\cdots,x_n)=$$
$$-f(0,-x_1,\cdots,-x_n)=-F(0,-x_1,\cdots,-x_n)$$

for all $(0,x_1,\cdots,x_n)\in S^{n-1}$. This contradicts an equivalent form of Borsuk-Ulam theorem which says that there does not exist any continuous function $l: B^n \mapsto S^{n-1}$ such that, $l(x)=-l(-x)$, for every, $x\in S^{n-1}$. □

Define the function $\theta: S^n \mapsto B^n$ by $\theta(x)=(0,x_1,\cdots,x_n)$. Given $|\bar{x}_0|<1$ and $y\in S^n$ and a function $f: S^n\mapsto S^n$, we define

$$\alpha(y)=\{x\in S^n | x_0=\bar{x}_0, \theta\circ f(\bar{x}_0,x_1,\cdots,x_n)=\beta\theta(y) \textit{ for some } \beta>0\}$$
$$A=\{y\in S^n | 1>y_0>\max\{f_0(x)|x\in\alpha(y)\}\}$$
$$B=\{y\in S^n | -1<y_0<\min\{f_0(x)|x\in\alpha(y)\}\}.$$

Theorem 1.8 *For any continuous function $f: S^n\mapsto S^n$ and for some given $|\bar{x}_0|<1$, if it satisfies that for each $(\bar{x}_0,x_1,\cdots,x_n)\in S^n$,*

$$f_0(\bar{x}_0,x_1,\cdots,x_n)\neq 1 \textit{ or } -1$$
$$\theta\circ f(\bar{x}_0,x_1,\cdots,x_n)=-k\theta\circ f(\bar{x}_0,-x_1,\cdots,-x_n)$$

for some $k>0$, then either $(\{e_0\}\cup A)\subset f(S^n)$ or $(\{-e_0\}\cup B)\subset f(S^n)$ or both.

Proof: Define the function $\phi: S^n\mapsto S^n$ by

$$\phi_0(x_0,x_1,\cdots,x_n)=\begin{cases}\bar{x}_0+(1-\bar{x}_0)x_0, & \textit{if} \quad x_0\geq 0\\ \bar{x}_0+(1+\bar{x}_0)x_0, & \textit{if} \quad x_0<0\end{cases}$$

and

$$\phi_i(x_0,x_1,\cdots,x_n)=\frac{x_i}{\sqrt{\phi_0^2(x)+\sum_{i=1}^n x_i^2}}, i=1,\cdots,n.$$

Clearly, ϕ is a continuous function. Next define the function $g: S^{n-1}\mapsto S^{n-1}$ by

$$g_i(0,x_1,\cdots,x_n)=\frac{f_i(\bar{x}_0,x_1\sqrt{1-\bar{x}_0^2},\cdots,x_n\sqrt{1-\bar{x}_0^2})}{\sum_{j=1}^n f_i^2(\bar{x}_0,x_1\sqrt{1-\bar{x}_0^2},\cdots,x_n\sqrt{1-\bar{x}_0^2})}, i=1,\cdots,n.$$

It follows from the assumption on f that g is continuous and $g(x)=-g(-x)$ for all $x\in S^{n-1}$. By Theorem 1.5, g is a surjective function. This implies that for any given $y\in S^n$ with $y\neq e_0$ or $-e_0$, there exists some $(\bar{x}_0,x_1,\cdots,x_n)\in S^n$ such that

$$\theta\circ f(\bar{x}_0,x_1,\cdots,x_n)=k\theta(y)$$

for some $k>0$. Now we construct two functions: The function $\eta_+: S^n\setminus\{e_0,-e_0\}\mapsto R$ is given by

$$\eta_+(x_0, x_1, \cdots, x_n) = max\,\{f_0(\bar{x}_0, y_1, \cdots, y_n) | \\ \theta \circ f(\bar{x}_0, y_1, \cdots, y_n) = k\theta(x_0, x_1, \cdots, x_n) \text{ for some } k > 0\};$$

The function $\eta_-: S^n \setminus \{e_0, -e_0\} \mapsto R$ is given by

$$\eta_-(x_0, x_1, \cdots, x_n) = min\,\{f_0(\bar{x}_0, y_1, \cdots, y_n) | \\ \theta \circ f(\bar{x}_0, y_1, \cdots, y_n) = k\theta(x_0, x_1, \cdots, x_n) \text{ for some } k > 0\}.$$

It can be verified that η_+ and η_- are continuous functions. Let $C = S^n \setminus (A \cup B \cup \{e_0\} \cup \{e_0\})$. Let the function $H: S^n \mapsto S^n$ be defined by

$$H(x) = \begin{cases} e_0, & \text{for} \quad x = e_0; \\ -e_0, & \text{for} \quad x = -e_0; \\ \dfrac{(0, x_1, \cdots, x_n)}{\sqrt{1 - x_0^2}}, & \text{for} \quad x \in C, \end{cases}$$

$$\begin{cases} H_0(x) = \dfrac{x_0 - \eta_+(x)}{1 - \eta_+(x)}, & \text{for} \quad x \in A; \\ H_i(x) = \dfrac{x_i}{H_0^2(x) + \sum_{i=1}^n x_i^2}, & \text{for} \quad x \in A, i = 1, \cdots, n; \end{cases}$$

$$\begin{cases} H_0(x) = \dfrac{x_0 - \eta_-(x)}{1 + \eta_-(x)}, & \text{for} \quad x \in B; \\ H_i(x) = \dfrac{x_i}{H_0^2(x) + \sum_{i=1}^n x_i^2}, & \text{for} \quad x \in B, i = 1, \cdots, n. \end{cases}$$

Define the function $F: S^n \mapsto S^n$ by

$$F(x) = H \circ f \circ \phi(x).$$

H, f, and ϕ are continuous, so is F. Moreover, for every $(0, x_1, \cdots, x_n) \in S^n$, we have

$$f \circ \phi(0, x_1, \cdots, x_n) \in C;$$
$$F(0, x_1, \cdots, x_n) = -F(0, -x_1, \cdots, -x_n).$$

It follows from Lemma 1.7 that either $S_+^n \subset F(S^n)$ or $S_-^n \subset F(S^n)$ or both. On the other hand, we know that

$H(\{e_0\} \cup A) = S_+^n$, and $H(\{-e_0\} \cup B) = S_-^n$.
This implies that either $(\{e_0\} \cup A) \subset f(S^n)$ or $(\{-e_0\} \cup B) \subset f(S^n)$ or both. □

2 Application to an Economic Model

Consider an exchange economy in which there are m agents (consumers or investors) and n

commodities or assets. Each agent is characterized by three parameters (ω^i, u^i, X^i), where ω^i is the initial endowment, $u^i: \mathbb{R}^n \mapsto \mathbb{R}$ is the direct or indirect utility function, and $X^i \subset \mathbb{R}^n$ is the feasible choice set. Let H denote the set of agents. For closely related models, we refer to Nielsen (1990), Polemarchakis and Siconolfi (1993). We impose the following conditions on the current model.

Assumption 1: *For each $h \in H$, the feasible choice set X^i, is either a compact and convex set containing ω^i in its interior, or a closed convex set bounded from below containing ω^i in its interior.*

Assumption 2: *For each $h \in H$, the utility function u^h is continuous and quasi-concave.*

Since the utility function u^h is only required to be quasi-concave, the satiation bundles of individual h may be a convex set. Define the price set $P = \{p \in \mathbb{R}^n \,|\, \sqrt{p \cdot p} = 1\}$.At a price vector $p \in \mathbb{R}^n$, the budget set of individual h is $B^h(p) = \{y \in X^h | \, p \cdot y \leq p \cdot \omega^i\}$.

The optimization problem of agent h at price p is to

maximize $u^h(y)$
ubject to $p \cdot y \leq p \cdot \omega^i$, $y \in X^i$.

The solution of this problem, $y^h(p), h \in H$, exists but need not be unique. Therefore, $y^h: P \Longrightarrow \mathbb{R}^n$ is usually a point-to-set mapping, and is called *the demand correspondence* of agent h. The aggregated excess demand correspondence $z: P \Longrightarrow \mathbb{R}^n$ is defined by

$$z = \sum_{h \in H} (y^h - \omega^h).$$

The following lemma is well-known; see for example Debreu (1959).

Lemma 2.1 *Under Assumptions 1 and 2, for every $h \in H$, the individual damand correspondence y^h, and the aggregated excess demand correspondence z are nonempty convex and compact valued, and are upper semi-continuous.*

Assumption 3: *There exists a vector $v \neq 0 \in \mathbb{R}^n$ such that there are no $p \in P$ and no $\alpha > 0$ with $\alpha v \in z(p)$.*

Definition 2.2 A competitive equilibrium is a pair $(p^*, y^{*h}, h \in H)$, of prices and consumption bundles, such that

$$y^{*h} \in y^h(p^*), \quad h \in H,$$
$$\textit{and}, \quad \sum_{h \in H} (y^{*h} - \omega^h) = 0.$$

Now we are ready to present our equilibrium existence theorem.

Theorem 2.3 *Under Assumptions 1, 2 and 3, there exists a competitive equilibrium in the economy.*

Proof: Suppose to the contrary that there exists no competitive equilibrium in the economy. That is, there is no price $p \in P$ such that $0 \in z(p)$. Let

$$Gr(y) = \{(p,q) | p \in P \text{ and } q \in z(p)\}$$

denote the graph of the aggregated excess demand correspondence z. It follows from Lemma 2.1 that z is closed and upper semi-continuous. We see that $Gr(z)$ is a closed set of $\mathbb{R}^{2n}$. Assumption 3 means that, for any $p \in P$ and $\alpha > 0$, $(p, \alpha v) \notin Gr(z)$.

We first prove that there is an $\epsilon > 0$ such that the distance $d((p,\alpha v), Gr(z)) > \epsilon$ for all $p \in P$ and $\alpha > 0$. Suppose not. Then there are two sequences $(q^k, \alpha^k v)$, and $(p^k, w^k) \in Gr(z)$, such that the distance $d((q^k, \alpha^k v),(p^k, w^k)) \to 0$. Without loss generality (if necessary, from the boundedness of the sequence we can choose a subsequence), assume that $p^k \to p$ and $w^k \to w$. From the upper semi-continuity of z, we see that $w \in z(p)$. And therefore $w \neq 0$. Next note that $\alpha^k v$ tends to W and that q^k tends to p. This means that there is some $\alpha > 0$ such that $\alpha v = w$. This contradicts Assumption 3 that there are no $p \in P$ and no $\alpha > 0$ with $\alpha v \in z(p)$.

Similarly, we prove that there is a $\delta > 0$ such that the distance $d((p,\alpha p), Gr(z)) > \delta$ for all $p \in P$ and $\alpha > 0$. Suppose not. Then there are also two sequences $(q^k, \alpha^k q^k)$, and $(p^k, w^k) \in Gr(z)$, such that the distance $d((q^k, \alpha^k q^k),(p^k, w^k)) \to 0$. Without loss of generality, we can also assume that $p^k \to p$, $w^k \to w$, and $w (\neq 0) \in z(p)$. Moreover, note that $q^k \to p$, and $\alpha^k q^k \to w$. Hence we have that there is an $\alpha > 0$ such that $\alpha p = w \in z(p)$. But it is impossible since $p \cdot w \leq 0$ for all $p \in P$ and $w \in z(p)$.

Finally, by the von Neumann's Approximation Lemma (see Border (1985)) for any $k > 0$ there is a continuous function $f^k : P \to \mathbb{R}^n$ such that $Gr(f^k) \subset N_{1/k}(Gr(z))$, where

$$N_{1/k}(Gr(z)) = \{(q,w) \in \mathrm{IR}^{2n} \,|\, d((q,w), Gr(z)) < 1/k\}$$

Thus we see that when $1/k < min\{\epsilon, \delta\}$ there are no $p \in P$ and no $\alpha > 0$ such that $\alpha v = f^k(p)$ or $\alpha p = f^k(p)$. It follows Theorem 1.1 that there is $p^{*k} \in P$ such that $f^k(p^{*k}) = 0$ for all k with $1/k < min\{\epsilon, \delta\}$. (More precisely, suppose that there is no $p \in P$ with $f^k(p) = 0$. Then we can normalize f^k on P. So we may assume that f^k is a continous function mapping from P into itself. Since f^k has no fixed point in P, it follows from Theorem 1.1 that f^k is a surjective function. This contradicts the fact that $v/(v \cdot v) \notin f^k(P)$.) Since P is a compact set containing the sequence $\{p^{*k}\}$, the sequence has a convergent subsequence. For simplicity, we may assume that $p^{*k} \to p^* \in P$. Recall that $Gr(z)$ is closed and $(p^{*k}, 0) = (p^{*k}, f^k(p^{*k})) \in N_{1/k}(Gr(z))$ for all k with $1/k < min\{\epsilon, \delta\}$. So, we must have that $0 \in z(p^*)$. This contradicts the hypothesis that there is no price $p \in P$ such that $0 \in z(p)$. We are done. □

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